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# DOWN-RANGE ANTI-MISSILE MEASUREMENT PROGRAM (DAMP)

## Microwave Emission from Plasma Electromagnetic Radiation: A Phenomenological Approach

Prepared for

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# ELECTROMAGNETIC RADIATION - A PHENOMENOLOGICAL APPROACH

G.G. Cloutier and W.B. Baker

## - A B S T R A C T -

This report treats the macroscopic approach to the problem of equilibrium radiation from plasmas based on the theory of electrical fluctuations and thermal radiation proposed by Rytov[1]. The basic ideas underlying this general theory as well as the connection with other known theories of equilibrium radiation are discussed.

The electromagnetic formulation of this theory is then presented. In particular the transition between the microscopic and the macroscopic electromagnetic equations is demonstrated and the connection with Rytov's equations is made. The statistical aspect of the problem is formulated and the general expressions for the time and space averages of the intensity of radiation is presented. Special attention is given to the derivation of the correlation function for dissipative media. Some examples of fluctuation phenomena are also worked out.

A general method for solving problems of equilibrium radiation is outlined. The case of a cylindrical plasma, which is an adequate model for the wake of a re-entry vehicle, is discussed based on a solution originally derived by Rytov. A detailed derivation of this solution as well as an alternate solution in terms of the power absorptivity of the cylinder is given in the Appendix.

# ELECTROMAGNETIC RADIATION - A PHENOMENOLOGICAL APPROACH

## - C O N T E N T S -

## FOREWORD

This report has been prepared to acquaint the reader with some of the results drawn from a study of the electromagnetic properties of plasmas, made by scientists (G. C. Clontier and W. B. Baker) associated with the RCA Victor, Ltd., Research Laboratories, Montreal, Canada. This work was performed under Contract DA-36-034-ORD-3144RD as part of the Down-range Anti-missile Measurement Program (DAMP).

ELECTROMAGNETIC RADIATION - A PHENOMENOLOGICAL APPROACH

## I INTRODUCTION

Rytov[1] has developed a phenomenological electromagnetic theory of thermal radiation which does not impose any restrictions upon the relationship between the wavelength of the emitted radiation and the size of the radiating body. This theory is sometimes referred to as the correlation theory of electromagnetic fluctuations. This theory includes as particular cases both the classical theory of thermal or equilibrium radiation, which treats thermal fluctuations in the geometrical optics region, as well as the theory of thermal noise in electrical circuits.

The problem of determining the thermal radiation emitted from a heated body (or equivalently, the electromagnetic fluctuation field) is reduced to a boundary-value problem. In particular, one looks for the solutions of the 'modified Maxwell equations' which satisfy certain appropriate boundary conditions. Rytov's modification of Maxwell's equations consists of the introduction of terms which represent random extraneous source-fields with zero radius of correlation, or equivalently, terms which represent random extraneous currents with zero radius of correlation. In either case these terms represent the distributed source of the incoherent electromagnetic radiation emitted by a body at a given temperature.

Rytov's theory forms the basis for:

(1) the evaluation of the passive electromagnetic radiation emitted by a plasma; and

(2) a generalised formulation of Kirchhoff's law which permits the equilibrium electromagnetic or thermal radiation spectrum of a body to be evaluated from a knowledge of its absorptivity spectrum for a body of any size and for any degree of absorptivity.

The latter result provides us with a very valuable method of treating problems involving the interaction of electromagnetic radiation of any wavelength with plasmas and other bodies, that is, problems of active radiation.

The theory of electrical fluctuations and thermal radiation as developed by Rytov was originated by Leontovich and Rytov[2] in a paper treating the influence of the skin effect on electrical noise. In this paper they based their ideas on theories advanced by Mandelshtam[3], who as early as 1907-8, was concerned with the relationship between the microscopic and the macroscopic descriptions of the electromagnetic properties of matter and with the importance of fluctuating phenomena in electrodynamics.

The quantities with which a physicist or engineer works and measures are in general, macroscopic quantities. Measurements involve lengths which are very large in comparison with interatomic distances and involves time intervals which are very long when compared with characteristic atomic time intervals, say, for example, the period of time for an electron to rotate about the nucleus. As such, macroscopic quantities are statistical space-time averages of the more fundamental but very rapidly varying microscopic quantities.

In passing from a microscopic to a macroscopic description one is, in general, content to work with mean linear values. There is no a priori evidence that this procedure is valid, that is, that the mean linear values will agree with the values obtained by experiment. Usually this procedure is experimentally justified.

The concept of a mean linear value includes the possibility of larger or smaller deviations, and single measurements can yield values which fluctuate to a greater or lesser extent about this mean value. The good agreement between the statistical mean values and the macrophysical experimental data may be interpreted to signify that the fluctuations (deviations from the average value) encountered in statistical considerations are, in cases of agreement, generally speaking very small. However there do exist many macroscopic effects which owe their origin to the existence of fluctuations which are not negligible, for example, Brownian motion,

Schottky effect, thermal noise, etc. Fluctuations are not included in mean values.

As a measure of the magnitude of fluctuations we cannot take the mean value of the fluctuations, for it is equal to zero because the mean value of a quantity is defined so as to make the deviations in both directions equally probable. A possible measure is given by the mean value of the square of the fluctuations.

It is customary to define the fluctuations as equal to the difference between the special measured values $E(n)$ and the mean value $\bar{E}$, that is, by $\Delta E = E(n) - \bar{E}$. The measure for fluctuations is therefore equal to $\overline{\langle \Delta E(n) \rangle^2} = \langle \overline{E(n) - \bar{E}} \rangle^2$. Since the formation of a mean is a linear process (and since any mean value is a constant with respect to further operations of taking a mean: $\bar{\bar{E}} = \bar{E}$), we have $\overline{\langle \Delta E(n) \rangle^2} = \overline{E^2} - \overline{2E\bar{E}} + \bar{E}^2 = \overline{E^2} - \bar{E}^2$. That is, the mean square of a fluctuating quantity is equal to the difference between the mean value of the square of the quantity and the square of the mean value. We note that if the mean value of a quantity is zero then the measure of fluctuations is equal to the mean value of the quantity, i.e. $\overline{\langle \Delta E(n) \rangle^2} = \overline{E^2}$.

Previous to Rytov's general theory of electromagnetic fluctuations there existed two theories which treated electromagnetic fluctuation phenomena; however one is limited to the

quasi-stationary region and the other to the geometrical-optics region of the electromagnetic spectrum. They are, respectively:

(1) Nyquist's theory of thermal noise in linear electrical circuits, and

(2) the classical theory of thermal or equilibrium radiation as developed by Kirchhoff, Stefan-Boltzmann, Clausius and Planck.

Because the concepts and quantities employed to describe the phenomena and also because the theoretical analysis is different, it is, perhaps, not immediately obvious that the underlying phenomena are the same. However a little thought makes it clear that the underlying phenomena are indeed the same.

The concepts and quantities which a physicist creates, in order to describe the state of a physical system, are in general, dictated by what is experimentally significant or measurable. In the case of thermal noise in linear electrical circuits we are interested in currents and potential differences across capacitors and resistors. In the case of thermal radiation the intensity and energy of the emitted or absorbed radiation is of interest.

While the physical systems involved in each of the above cases are quite different, both phenomena are macrophysical

electromagnetic phenomena and as such are governed by Maxwell's equations. Now in general, the solution of Maxwell's equations with appropriate boundary conditions is a difficult problem.

However if the rate of change of the electromagnetic field is not too large (wavelength much greater than a characteristic length of the physical system) we can characterise the system, as far as its electromagnetic properties are concerned, by lumped parameters such as resistors, capacitors, inductors, etc. In this case it is the currents in the conductors and the potential differences across capacitors and resistors which are of experimental and theoretical interest.

If on the other hand the wavelength of the variable fields is very small when compared with a characteristic length of the physical system we can make use of the concepts of geometric optics. In this case we can talk about the path of an electromagnetic wave and work with such quantities as the intensity and energy of the radiation.

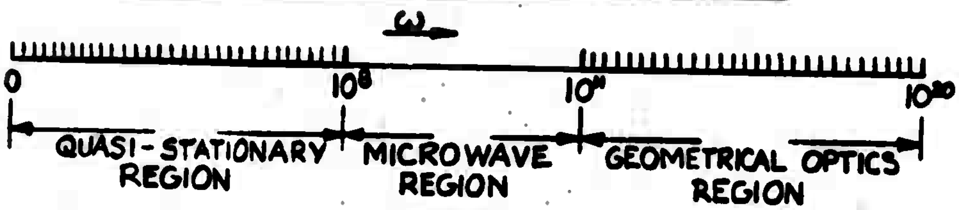
In both cases a theoretical simplification is achieved. In the following we show that if the emphasis is placed on the electromagnetic aspect of the phenomena, in particular on the fields $\vec{E}$ and $\vec{H}$, then the mean density of equilibrium radiation energy itself is a measure of the intensity of fluctuation and also that we can attribute thermal radiation phenomena to fluctuation phenomena. Hence we can regard

both the problem of thermal noise in linear circuits and the classical problem of thermoradiation as having a common basis. Maxwell's equations govern the behaviour of all macroscopic electromagnetic phenomena. They are:

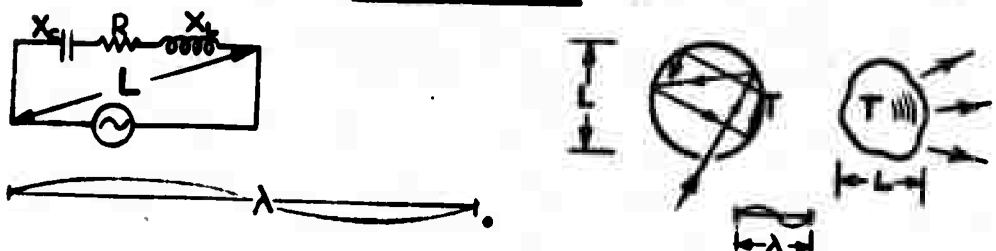
$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \quad , \qquad \nabla \times \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} + \frac{4\pi \vec{J}}{c} \ ,$$

$$\nabla \cdot \vec{D} = 4\pi \rho_t \quad , \qquad \nabla \cdot \vec{B} = 0 \ .$$

**ELECTROMAGNETIC RADIATION FREQUENCY SPECTRUM**




**Physical System**




Linear Electrical Circuit | Black Body Radiation

**Nature of the Electromagnetic Radiation**

(1) $\lambda \gg L$ (2) $\lambda \ll L$

## Resulting Simplification

Because $\lambda \gg L$ we can characterise the electromagnetic properties of the above physical system by lumped parameters such as resistors, inductors, and capacitors, etc.

Because of $\lambda \ll L$ we can introduce the concept of rays, that is the electromagnetic radiation travels along well-defined paths.

## Quantities of Interest

Current in conductors, ($j$). Potential difference, ($V$), across resistors, capacitors, etc.

Intensity of Radiation, $I_\nu$
Absorption and emission coefficients $a_\nu$, $e_\nu$.
Spectral distribution of radiation. $u = u(\nu, T)$
Average energy:

$$\bar{u} = \frac{1}{8\pi} \int (\overline{E^2} + \overline{H^2}) dv .$$

## Problem of Interest

Thermal Noise. (Nyquist)

The theory is formulated as a Fluctuation problem.

Equilibrium Radiation. Clausius, Kirchhoff, and Planck. The problem is treated as one in Phenomenological Thermodynamics.

## Basic Assumption

Assumes there exists a random Emf which produces fluctuations of the current and voltage in the circuit.

Assumes there exists a thermal motion of the charges in the body which results in the production of an incoherent radiation field.

In both cases the macroscopic or space-time average electromagnetic properties of the physical system are given by the dielectric coefficient and the magnetic permeability.

The measure of the fluctuations in the quantities of interest are:

$$(\Delta j)^2 = \overline{j^2} - \overline{j}^2 ,$$

$$(\Delta V)^2 = \overline{V^2} - \overline{V}^2 .$$

Because the circuit is linear, we have:

$j = \sigma E$ and $V = E \cdot d$ .

Therefore $\overline{j} \propto \overline{E}$ and $\overline{V} \propto \overline{E}$ .

For electric circuits which are in thermal equilibrium we have that

$$\overline{E} = 0 .$$

Hence the fluctuations of the quantities of interest in a linear electric circuit are proportional to the mean values of the square of the field E.

$$(\Delta j)^2 \propto \overline{E^2} , \quad (\Delta V)^2 \propto \overline{E^2} .$$

The problem of thermal noise in linear electric circuits can be traced back to the fluctuations of the electromagnetic field.

An insight into Nyquist's approach - in particular, the assumption of the existence of a random Emf in the circuit which produces fluctuations in the current and voltage--can be found by considering the problem of stationary currents. A stationary current is impossible in a purely irrotational electric field, since in a stationary current energy is expended at a rate

Since we are interested here in the average energy

$$\overline{u} = \frac{1}{8\pi} \int (\overline{E^2} + \overline{H^2}) \, dv .$$

and since for thermal equilibrium we have $\overline{E}$ and $\overline{H}$ equal to zero, we can regard the problem of classical equilibrium radiation as a fluctuation problem by regarding the average energy as a measure of the fluctuation of the thermal motion of the charges.

$\vec{j} \cdot \vec{E}$ per unit volume, and this energy cannot be provided by an irrotational field. Stationary currents are possible only if there are present sources of electric field known as electromotive forces, which produce nonirrotational fields. Hence one assumes the existence of such an Emf. It now becomes possible to have stationary currents and to explain in electrical terms various forces which affect the charges of the system under consideration but which are determined by sources not pertaining to this system. These forces operate at the expense of outside sources of energy - mechanical, chemical, etc.

All physical systems find themselves in a radiation field, i.e. an electromagnetic field in which are present both radiation in the optical frequency region as well as radiation in the lower frequencies. According to Nyquist's approach the presence of this radiation in linear circuits gives rise to random Emf's. According to the classical theory of equilibrium radiation the presence of this radiation gives rise to the thermal motion of charges. We can achieve a unification by replacing the random Emf's by the concept of thermal motion of charges in linear circuits.

We can then say that the existence of a thermal radiation field as well as the existence of thermal noise in linear electric circuits is due to the thermal motion of the charges in the system.

It is now clear that to produce a general theory of electromagnetic fluctuations due to the thermal motion of the charges in a body we must make use of Maxwell's equations but that we cannot use them directly. This is so because the field quantities $\vec{E}$ and $\vec{H}$ which appear in Maxwell's equations represent average or mean values; any fluctuations in the microscopic fields as well as in the microscopic charge and current densities have been averaged out.

By studying the method of obtaining Maxwell's equations from the more fundamental microscopic (Lorentz) equations (Maxwell's equations for vacuum) we can get an insight as to how Maxwell's equations need to be modified so as to include in them terms which represent the thermal motion of charges.

# II ELECTRODYNAMIC FORMULATION

## 2.1 General Remarks

Maxwell's equations govern the behaviour of large-scale electromagnetic phenomena and for this reason they are often referred to as the "macroscopic field equations". Maxwell's equations do not directly take into account the atomicity of matter and electricity, for at the time when he developed his equations (1861-73) the electrical origin of matter was not known.

Maxwell's equations are:

$$\text{(a)}\quad \nabla\times\vec{H} = \frac{1}{c}\frac{\partial\vec{D}}{\partial t} + \frac{4\pi}{c}\vec{J}\quad,\qquad \text{(b)}\quad \nabla\cdot\vec{D} = 4\pi\rho_t\quad,\qquad (2.1)$$

$$\text{(c)}\quad \nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\quad,\qquad \text{(d)}\quad \nabla\cdot\vec{B} = 0$$

They are four in number and involve six quantities $\vec{E}$, $\vec{D}$, $\vec{B}$, $\vec{H}$, $\rho$ and $\vec{J}$. Therefore in order to determine the six quantities uniquely we need to have two more equations at our disposal. The additional two equations are the so-called 'constitutive relations':

$$\varepsilon = \frac{\vec{D}}{\vec{E}}\quad,\qquad \mu = \frac{\vec{B}}{\vec{H}}$$

which may be regarded as defining the dielectric constant $\varepsilon$ and the magnetic permeability $\mu$. The latter equations represent our experimental knowledge of the physical properties of the medium.

When experiment brought to light the electrical origin of matter, i.e. that matter consisted of charged particles, nuclei and electrons, Lorentz felt that by probing down to sub-atomic distances it should be possible to formulate the equations of electrodynamics in terms of charges in vacuum without the introduction of ponderable dielectric and magnetic media. With this in mind Lorentz proposed and studied what are now called the "microscopic field equations":

$$\text{(a)} \quad \nabla \times \vec{h} = \frac{1}{c}\frac{\partial \vec{e}}{\partial t} + \frac{4\pi\rho\vec{v}}{c} \quad , \qquad \text{(b)} \quad \nabla \cdot \vec{e} = 4\pi\rho \ ,$$

$$\text{(c)} \quad \nabla \times \vec{e} = -\frac{1}{c}\frac{\partial \vec{h}}{\partial t} \quad , \qquad \text{(d)} \quad \nabla \cdot \vec{h} = 0 \ . \qquad (2.2)$$

These are just Maxwell's equations for vacuum.

This program which is known as the classical theory of electrons consists in trying to explain all observable phenomena in terms of the microscopic field equations. For example one task of the microscopic theory is to explain the dielectric constant and magnetic permeability, already measured macroscopically, in terms of atomic concepts.

In short Lorentz regarded matter and the atoms constituting the matter as consisting of positive and negative charges, coupled in some way and moving in vacuum, and the electromagnetic field produced by these charges. There is only one class of field vector

namely that corresponding to the vacuum; these are denoted by $\vec{e}$ and $\vec{h}$. In addition there exists only one kind of electric current, the convection current which results from the motion of the charges. We have:

$\vec{e}$ electric field intensity , $\vec{h}$ magnetic field intensity,

$\rho$ charge density, , $\vec{v}$ velocity of the charges.

These quantities are to satisfy the microscopic field equations. For vacuum ($\rho = 0$) these equations are identical to Maxwell's equations. Outside of matter we can at once identify the vectors $\vec{e}$ and $\vec{h}$ with the macroscopic field vectors $\vec{E}$ and $\vec{H}$, respectively. However special care is required in interpreting the significance of $\vec{e}$ and $\vec{h}$ in matter or even inside of atoms.

Maxwell's equations characterize the state of a dielectric between the plates of a charged capacitor by two vectors $\vec{E}$ and $\vec{P}$ and the dielectric constant $\varepsilon$. The description of the same body by means of the field vectors $\vec{e}$ and $\vec{h}$ is infinitely more complicated. The $\vec{e}$ and $\vec{h}$ vectors are rapidly varying functions of space and time while the field vectors of Maxwell's equations are assumed to be constant (or so slowly varying that they are accessible to measurement) in "physically infinitesimal" volume elements of the body under consideration.

"Physically infinitesimal" volume elements are those large enough to contain many atoms or molecules but which at the same time

are very small with respect to the size of the bodies being considered. Maxwell's macroscopic field quantities $\vec{E}$ and $\vec{H}$ are to be regarded as the space-time mean values of the microscopic field vectors $\vec{e}$ and $\vec{h}$. Maxwell's equations in their application to matter possess only a limited validity (e.g. $\varepsilon$ and $\mu$ are macro-parameters) however the equations of Lorentz' electron theory - in their coupling of the fields to the charge and current densities - are strictly valid. Of course, it is necessary to make special assumptions, depending on the nature of the matter, about the distribution of electrons in atoms in order to apply the microscopic equations to the interior of matter.

In the following we give an analysis of the macroscopic equations from the microscopic or atomic point of view.

We cannot immediately relate the macroscopic equations (2.1) to the microscopic equations (2.2) since the quantities in (2.2) are rapidly varying functions of space and time while the quantities in (2.1) are slowly varying functions of space and time. We must first average the microscopic quantities over physically infinitesimal regions.

## 2.2 Space-time Average Procedure

Consider any space-time point $P(x_1, x_2, x_3, x_4)$ and a neighboring space-time point $P'(x_1', x_2', x_3', x_4')$. We define the space-time average value of any microscopic quantity $F$ at the point $P$ as

$$\bar{F}(x_1,x_2,x_3,x_4) = \frac{\iiiint F(x_1', x_2', x_3', x_4')\, dx_1'\, dx_2'\, dx_3'\, dx_4'}{\iiiint dx_1'\, dx_2'\, dx_3'\, dx_4'}$$

or

$$\bar{F}(P,t) = \frac{1}{V'\cdot T}\int_V dv' \int_T F(P',t')\, dt'$$

where we are integrating over the space-time region $x_1', x_2', x_3', x_4'$ about the fixed point $x_1,x_2,x_3,x_4$. We wish to show:

$$\overline{\frac{\partial F}{\partial x_\alpha}} = \frac{\partial \bar{F}}{\partial x_\alpha} \qquad \text{where } \alpha = 1, 2, 3, 4 .$$

Let: $$x_\alpha' - x_\alpha = \xi_\alpha ,$$

then $$dx_1'\, dx_2'\, dx_3'\, dx_4' = d\xi_1\, d\xi_2\, d\xi_3\, d\xi_4 = d\Omega .$$

Therefore we have

$$\bar{F}(x_1,x_2,x_3,x_4) = \frac{1}{\Omega}\iiiint F(x_1+\xi_1 , x_2+\xi_2 , x_3+\xi_3 , x_4+\xi_4)\, d\Omega ,$$

where now in the right hand side the coordinates of the fixed point P appear as parameters. We have finally:

$$\frac{\partial}{\partial x_\alpha} F(x_1+\xi_1 , x_2+\xi_2 ,\ldots) = \frac{\partial}{\partial \xi_\alpha} F(x_1+\xi_1 , x_2+\xi_2 , \quad ) = \frac{\partial}{\partial x_\alpha'} F(x_1',x_2',x_3',x_4')$$

$$\frac{\partial}{\partial x_\alpha} \bar{F}(x_1,x_2,x_3,x_4) = \frac{1}{\Omega}\int \frac{\partial}{\partial x_\alpha'} F(x_1', x_2', x_3', x_4')\, d\Omega = \overline{\frac{\partial}{\partial x_\alpha} F(x_1,x_2,x_3,x_4)} .$$

That is, differentiation and averaging may be interchanged. Hence we have for any linear differential equation $D(F)$:

$$\overline{D(F)} = D(\overline{F})$$

We can now take the average of both sides of our microscopic equations and use the above relation to replace the averages of the derivatives of the field quantities by the derivatives of the average of the field quantities. We have

$$\text{(a)} \quad \nabla \times \overline{\vec{h}} = \frac{1}{c}\frac{\partial}{\partial t}\overline{\vec{e}} + \frac{4\pi}{c}\,\overline{\rho\vec{v}} \,, \qquad \text{(b)} \quad \nabla \cdot \overline{\vec{e}} = 4\pi\,\overline{\rho} \,, \qquad (2.3)$$

$$\text{(c)} \quad \nabla \times \overline{\vec{e}} = -\frac{1}{c}\frac{\partial \overline{\vec{h}}}{\partial t} \,, \qquad \text{(d)} \quad \nabla \cdot \overline{\vec{h}} = 0 \,.$$

Since neither $\overline{\rho}$ nor $\overline{\rho\vec{v}}$ occur in equations (2.3c ) and (2.3d ) we can immediately make the following identifications:

$$\overline{\vec{e}} = \vec{E} \qquad \text{and} \qquad \overline{\vec{h}} = \vec{B} \,. \qquad (2.4)$$

On substituting these relations in equations (2.3a ) and (2.3b ) we get:

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\,\overline{\rho\vec{v}} \,, \qquad (2.5)$$

$$\nabla \cdot \vec{E} = 4\pi\,\overline{\rho} \,. \qquad (2.6)$$

These equations are to be compared with the macroscopic equations:

$$\nabla \times \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} + \frac{4\pi}{c}\vec{J} \qquad \text{and} \qquad \nabla \cdot \vec{D} = 4\pi\rho_t \,.$$

## 2.3 Transition from Microscopic to Macroscopic Equations

At this point we can proceed in two ways[4,5]. Since neither of the two ways appears to be well-known we outline both methods.

### (1) Method A

If, in equation (2.2a) and (2.2b), we introduce the electric and magnetic polarisation vectors $\vec{P}$ and $\vec{M}$ (the electric and magnetic moments per unit volume, respectively) by

$$\vec{H} = \vec{B} - 4\pi \vec{M} \qquad \text{and} \qquad \vec{D} = \vec{E} + 4\pi \vec{P} \tag{2.7, 2.8}$$

we obtain

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}[\vec{J} + \dot{\vec{P}} + c\nabla \times \vec{M}] , \tag{2.9}$$

$$\nabla \cdot \vec{E} = 4\pi [\rho_t - \nabla \cdot \vec{P}] . \tag{2.10}$$

We will be able to say that Maxwell's equations follow from the microscopic equations if it can be shown that, in the sense of our definitions of average values we have

$$\overline{\rho \vec{v}} = \vec{J} + \dot{\vec{P}} + c\nabla \times \vec{M} = \vec{j}_c + \vec{j}_p + \vec{j}_m , \tag{2.11}$$

$$\overline{\rho} = \rho_t - \nabla \cdot \vec{P} = \rho_t - \rho_p . \tag{2.12}$$

We first calculate the conduction and polarisation currents $\vec{j}_c$ and $\vec{j}_p$, respectively and then the magnetisation current $\vec{j}_m$.

In attempting to describe the processes occurring in a material body we consider the body as an assembly of neutral atoms, neutral molecules, ions and free electrons. The simplest process contributing to the transport of electricity is due to the motion of the ions and free electrons. An electric field or a charge density gradient will give rise to a directed motion of the charged particles. This gives rise to the conduction current $\vec{j}_c$. The ions and free electrons also contribute to the average charge density; this contribution corresponds to what is called in Maxwell's theory the true charge density $\rho_t$.

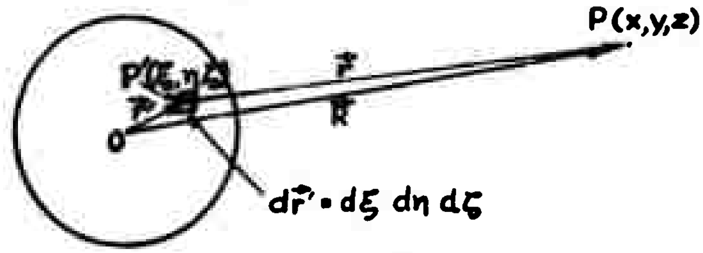
We now consider the polarisation current $\vec{j}_p$. This current arises from the contributions of neutral atoms and molecules. In Maxwell's theory the polarisation $\vec{P}$ is defined as the dipole moment per unit volume. It arises from the vector sum of the dipole moments of the individual atoms and molecules. Let $\vec{p}_i$ be the dipole moment of the i'th molecule and let there be N molecules per unit volume. Then the polarisation vector can be written as

$$\vec{P} = \sum_{i=1}^{N} \vec{p}_i \quad . \tag{2.13}$$

Let the charge and current distributions in a molecule be given by $\rho$ and $\rho\vec{v}$, respectively. The resulting electromagnetic field can be characterised by a scalar and a vector potential given by:

$$\phi(x,y,z,t) = \iiint \frac{\rho(\xi,\eta,\zeta,t-\frac{r}{c})}{r}\, d\xi\, d\eta\, d\zeta , \tag{2.14}$$

$$\vec{A}(x,y,z,t) = \iiint \frac{[\rho\vec{v}]\,(\xi,\eta,\zeta,t-\frac{r}{c})}{rc}\, d\xi\, d\eta\, d\zeta , \tag{2.15}$$




Schematic diagram for evaluating $\phi$ and $\vec{A}$.

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2} , \tag{2.16}$$

$$\vec{r} = \vec{R} - \vec{r}' . \tag{2.17}$$

Take the origin to be located at the center of the molecule and consider the field at a point for which $|\vec{R}| >>> |\vec{r}'|$ $(R >>> d)$. Then we can make the approximations

$$r = \sqrt{R^2 + r'^2 - 2\vec{r}'\cdot\vec{R}} \approx R - \frac{(\vec{r}'\cdot\vec{R})}{R} , \tag{2.18}$$

$$\frac{1}{r} \approx \frac{1}{R}\,\frac{1}{1-\frac{\vec{r}'\cdot\vec{R}}{R^2}} \approx \frac{1}{R} + \frac{(\vec{r}'\cdot\vec{R})}{R^3} + \ldots , \tag{2.19}$$

and write

$$\rho(\vec{r}\,',t-\frac{r}{c}) \approx \rho(\vec{r}\,',t-\frac{R}{c}+\frac{\vec{r}\,'\cdot\vec{R}}{Rc})$$

$$\approx \rho(\vec{r}\,',t-\frac{R}{c}) + \frac{(\vec{r}\,'\cdot\vec{R})}{Rc}\,\dot{\rho}(\vec{r}\,',t-\frac{R}{c}) + \dots \qquad (2.20)$$

Hence we can write

$$\phi \approx \frac{1}{R}\iiint \rho\, d\vec{r}\,' + \frac{1}{R^3}\iiint \rho(\vec{r}\,',t-\frac{R}{c})(\vec{r}\,'\cdot\vec{R})\, d\vec{r}\,'$$

$$+ \frac{1}{R^2c}\iiint (\vec{r}\,'\cdot\vec{R})\dot{\rho}\, d\vec{r}\,' + \dots \qquad (2.21)$$

$$\vec{A} = \frac{1}{Rc}\iiint \rho\vec{v}\, d\vec{r}\,' + \frac{1}{R^3c}\iiint \rho\vec{v}\,(\vec{R}\cdot\vec{r}\,')\, d\vec{r}\,'$$

$$+ \frac{1}{R^2c^2}\iiint (\vec{R}\cdot\vec{r}\,')\frac{\partial}{\partial t}(\rho\vec{v})\, d\vec{r}\,' + \dots \qquad (2.22)$$

where $d\vec{r}\,' = d\xi\, d\eta\, d\zeta$ .

By using the following identities and carrying out a partial integration we can rewrite the right hand sides of the expressions for $\phi$ and $\vec{A}$ :

$$\rho v_\xi + \xi\nabla\cdot(\rho\vec{v}) = \frac{\partial}{\partial\xi}(\rho v_\xi \xi) + \frac{\partial}{\partial\eta}(\rho v_\eta \xi) + \frac{\partial}{\partial\zeta}(\rho v_\zeta \xi)\,, \qquad (2.23)$$

$$\rho v_\xi \xi + \rho v_\xi \xi + \xi^2\nabla\cdot(\rho\vec{v}) = \frac{\partial}{\partial\xi}(\rho v_\xi \xi^2) + \frac{\partial}{\partial\eta}(\rho v_\eta \xi^2)$$

$$+ \frac{\partial}{\partial\zeta}(\rho v_\zeta \xi^2)\,, \qquad (2.24)$$

$$\rho v_\xi \eta + \rho v_\eta \xi + \xi\eta\nabla\cdot(\rho\vec{v}) = \frac{\partial}{\partial\xi}(\rho v_\xi \xi\eta) + \frac{\partial}{\partial\eta}(\rho v_\eta \xi\eta) + \frac{\partial}{\partial\zeta}(\rho v_\zeta \xi\eta). \qquad (2.25)$$

e.g. Consider the x-component of $\int \rho\vec{v}\, d\vec{r}'$, that is, $\iiint \rho v_\xi \, d\xi\, d\eta\, d\zeta$.

Using the first identity we obtain

$$\iiint \rho v_\xi \, d\xi\, d\eta\, d\zeta = -\iiint \xi\nabla\cdot(\rho\vec{v})d\xi\, d\eta\, d\zeta + \iiint \frac{\partial}{\partial\xi}(\rho v_\xi \xi)\, d\xi\, d\eta\, d\zeta + \iiint \frac{\partial}{\partial\eta}(\rho v_\eta \xi)d\eta\, d\xi\, d\zeta + \iiint \frac{\partial}{\partial\zeta}(\rho v_\zeta \xi)d\zeta\, d\xi\, d\eta .$$

Carrying out a partial integration in the last three terms we are left with a surface integral which can be made to vanish since we can move the surface outside the molecule. We are left with the relation

$$\iiint \rho v_\xi \, d\xi\, d\eta\, d\zeta = -\iiint \xi\nabla\cdot(\rho\vec{v})d\xi\, d\eta\, d\zeta$$

or in vector notation

$$\int \rho\vec{v}\, d\vec{r}' = -\int \vec{r}'(\nabla\cdot\rho\vec{v})\, d\vec{r}'. \qquad (d\vec{r}' = d\xi\, d\eta\, d\zeta)$$

Finally using the equation of continuity, $\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0$, we obtain

$$\int \rho\vec{v}\, d\vec{r}' = \int \vec{r}'\dot{\rho}\, d\vec{r}' .$$

Proceeding in a similar fashion for the other terms in $\phi$ and $\vec{A}$, we obtain

$$\phi = \frac{1}{R^3}\int \rho[\vec{r}'\cdot\vec{R}]d\vec{r}' + \frac{1}{R^2c}\int \vec{r}'\dot{\rho}\, d\vec{r}' = \left[\frac{(\vec{p}\cdot\vec{R})}{R^3} + \frac{(\dot{\vec{p}}\cdot\vec{R})}{R^2c}\right]_{t-\frac{R}{c}}, \qquad (2.26)$$

$$\vec{A} = \left[\frac{\dot{\vec{p}}}{cR} - \frac{\vec{R}\times\vec{m}}{R^3} - \frac{\vec{R}\times\vec{m}}{cR^2}\right]_{t-\frac{R}{c}}, \qquad (2.27)$$

where we have defined the electric and magnetic dipole moments for a continuous distribution of charge by

$$\vec{p} = \int \rho\vec{r}'\, d\vec{r}', \qquad (2.28)$$

$$\vec{m} = \frac{1}{2c}\int (\vec{r}\times\rho\vec{v})\, d\vec{r}'. \qquad (2.29)$$

The resulting expressions for the scalar potential $\phi$ and the vector potential $\vec{A}$ are just those which would occur if we were to consider the scalar and vector potential for an electric dipole of dipole moment $\vec{p}$ and a magnetic dipole with dipole moment $\vec{m}$. That is, we can regard the molecule, as far as its field is concerned, as consisting of an electric dipole and a magnetic dipole.

If we wish to replace the continuous distribution of charge by a discrete one we need merely to redefine $\vec{p}$ and $\vec{m}$ as follows:

$$\vec{p} = \sum_i e_i\, \vec{r}_i \qquad (2.30)$$

and

$$\vec{m} = \frac{1}{2c}\sum_i \vec{r}_i \times e_i\, \vec{v}_i \qquad (2.31)$$

(that is, replace the integration by a summation over all the discrete charges $e_i$ in a given molecule).

We assume that the total number of neutral molecules per unit volume is N and let these consist of:

$n_1$ molecules of type 1,

$n_2$ molecules of type 2,

.

.

$n_\alpha$ molecules of type $\alpha$.

We have $N = n_1 + n_2 + n_3 + \ldots$ . We denote the electric moment of a molecule of type $\alpha$ by $\vec{p}_\alpha = \sum_i e_{\alpha i} \vec{r}_{\alpha i}$ where the summation over the index i implies a summation over all the discrete charges $e_i$ constituting a given molecule of type $\alpha$. We consider the change in the dipole moment of each molecule which results from the displacement of the first electron through a distance $\delta\vec{r}_1$, the second electron through a distance $\delta\vec{r}_2$ and so forth. Let the displacement $\delta\vec{r}_i$ of the i'th electron have components $\xi_i$, $\eta_i$, $\zeta_i$ in the x- , y- and z-directions, respectively. The total charge (belonging to molecules of type $\alpha$) which passes through a surface element dS whose normal is parallel to the x-axis is

$$n_\alpha \sum_i e_{\alpha i}\, \delta\xi_{\alpha i}\, dS \; .$$

If $\vec{v}_{\alpha i}$ is the velocity of the i'th electron we have the relation $\delta\dot{\vec{r}}_{\alpha i} = \vec{v}_{\alpha i}$ . Therefore a change in the dipole moment $\vec{p}$ results in a

current with components

$$j_{\alpha x} = n_\alpha \sum_i e_{\alpha i} \dot{\xi}_{\alpha i} ,$$

$$j_{\alpha y} = n_\alpha \sum_i e_{\alpha i} \dot{\eta}_{\alpha i} ,$$

$$j_{\alpha z} = n_\alpha \sum_i e_{\alpha i} \dot{\zeta}_{\alpha i} ,$$

in the x-, y-, and z-directions respectively. That is, a change in the dipole moment $\vec{p}$ of a molecule of type $\alpha$ results in a current $\vec{j}_\alpha = n_\alpha \Sigma\, e_{\alpha i} \dot{\vec{r}}_{\alpha i} = n_\alpha \dot{\vec{p}}_\alpha$. Summing over all types of molecules we find the total polarization current $\vec{j}_p$ to be

$$\sum_\alpha n_\alpha \dot{\vec{p}}_\alpha = \dot{\vec{P}} , \tag{2.32}$$

where $\vec{P} = n_1\vec{p}_1 + n_2\vec{p}_2 + \dots n_\alpha\vec{p}_\alpha + \dots$, is the total electric dipole moment per unit volume.

It immediately follows from this result that a divergence of $\vec{P}$ must be associated with the average charge density $\bar{\rho}$ even if there are no ions or free electrons present. This is the contribution to the average charge density which arises from the presence of neutral atoms and molecules. A change of magnitude dP in the polarization $\vec{P}$ results in a charge of magnitude $dSdP_n$ passing out of a volume element dv. From the conservation of charge we can write

$$d\left[\iiint \rho\, dv\right] = -\iint dP_n\, dS = -\iint \frac{\partial P_n}{\partial t}\, dt\, dS$$

or

$$\frac{d}{dt}\iiint \rho\, dv = -\iint \frac{\partial P_n}{\partial t}\, dS\ ,$$

where $\rho$ denotes the density of charge, $dP_n$ denotes the change in $\vec{P}$ in the direction $\vec{n}$, where $\vec{n}$ is a unit vector perpendicular to the surface element $dS$. If $P = 0$ for $\rho = 0$ we have

$$\iiint \rho\, dv = -\iint P_n\, dS\ ,$$

and on using Gauss's divergence theorem we obtain the following result

$$\rho_p = -\nabla\cdot\vec{P}\ , \qquad (2.33)$$

where the index p implies a charge density due to the polarisation.

The total average charge density can now be written as a sum of the true charge density, $\rho_t$, that is the charge arising from the presence of ions and free electrons, and the polarisation charge density, $\rho_p$:

$$\bar{\rho} = \rho_t + \rho_p = \rho_t - \nabla\cdot\vec{P}\ . \qquad (2.34)$$

We now proceed to account for the third term in the expression for the average current density

$$\overline{\rho\vec{v}} = \vec{j}_e + \dot{\vec{P}} + c\nabla\times\vec{M}\ . \qquad (2.35)$$

According to the classical theory of electrons a magnetic field has its origin in the movement of charges. For the explanation of the magnetic moment $\vec{m}$ of an atom or molecule we must assume that the charges in the atom are in motion. We can consider these charges as point charges $e_i$ moving with velocities $\vec{v}_i$ or as a position-dependent current density $\vec{j}$.

We again assume that the total number of atoms per unit volume is N and let these consist of

$n_1$ atoms of type 1 ,

$n_2$ atoms of type 2 ,

.

.

$n_\alpha$ molecules of type $\alpha$.

We have $N = n_1 + n_2 + \dots + n_\alpha + \dots$ . The magnetisation or the total magnetic moment per unit volume is given by $\vec{M} = n_1\vec{m}_1 + n_2\vec{m}_2 + \dots$ .

Consider a "stationary current" $\vec{J}$ in a plane loop enclosing an area A. Let a be the cross-sectional area of the loop and let dl be an element of length of the loop. Then we have $\vec{j}\,dv = \vec{j}a\,dl = J\,\vec{dl}$. From the definition of magnetic moment

$$\vec{m} = \frac{1}{2c}\int \vec{r} \times \vec{j}\,dv ,$$

we have

$$\vec{m} = \frac{J}{2c}\oint \vec{r} \times \vec{dl} = \frac{J\vec{A}}{c} .$$

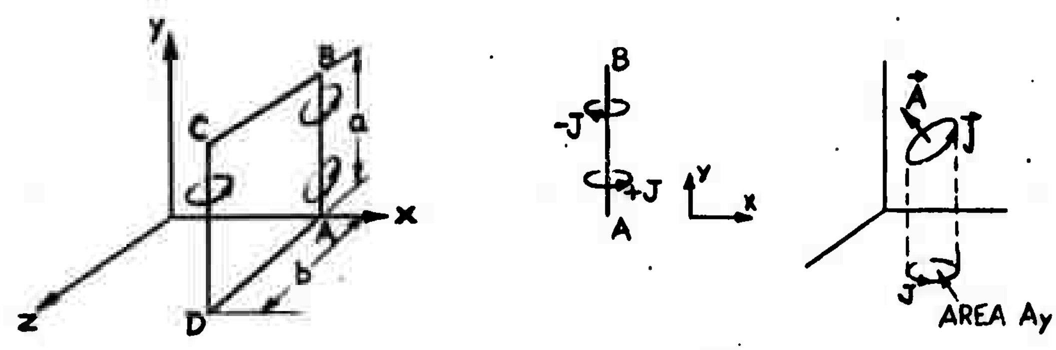
The magnetic moment of an atom due to a current density $\vec{j}$ or equivalently due to the quasi-stationary motion of the constituent electrons is given by

$$\vec{m} = \frac{1}{2c} \int \overline{\vec{r} \times \vec{j}}\, dv \;, \qquad (2.36)$$

$$\vec{m} = \frac{1}{2c} \sum_i e_i \,(\overline{\vec{r}_i \times \vec{v}_i}) \qquad (2.37)$$

respectively. The horizontal bar indicates a time average over the various orbits of the circulating electrons.

In order to see how the space-time average of a current density in an atom or molecules gives a contribution $\vec{j}_m$ to the average current density $\overline{\rho\vec{v}}$ we will consider a very simplified model. We assume that the current loops in an atom consist of equal plane ring currents of area A and magnitude J. Let there be n such current loops per unit volume. We now proceed to calculate the total current which threads a rectangle whose plane-area lies in the yz-plane and whose sides are of lengths a and b in the y- and z-directions, respectively. We only obtain a contribution to the total resulting current $\overline{j_x}$ if the current loops are threaded by the sides of the rectangle.




Current loops for evaluation of $\vec{j}_m$.

Consider the atomic current loops which are threaded by the side AB. We assume that the y-components of $\vec{m}$

$$m_y = \frac{J}{c} A_y$$

of all the atoms are the same. We now only need to find the number of atoms which have their current loops threaded by $\overline{AB}$. Consider a cylinder of length a and cross-section 1 $cm^2$ whose axis lies along $\overline{AB}$. This volume contains Na atoms. The probability that any current loop in this cylinder is threaded by $\overline{AB}$ is given by the ratio of the y-component area of the current loop and the cross-sectional area of the cylinder, that is, by $\frac{A_y}{1}$. If the $A_y$-areas vary we need only to average over the different values. In the length $\overline{AB}$ we have a contribution

$$J\,n\,a\,A_y = n\,a\,c\,m_y = a\,c\,M_y\Big]_{z=0}$$

to the current $\overline{j_x}$, where $M_y$ is the y-component of magnetisation, $\vec{M}$. We apply a similar analysis to the other three sides and find that: the atoms threaded by $\overline{DC}$, $\overline{AD}$, and $\overline{BC}$ gives contributions

$$-a\,c\,M_y\Big]_{z=b} \quad , \quad -b\,c\,M_z\Big]_{y=0} \quad , \quad +b\,c\,M_z\Big]_{y=a}$$

respectively, to the total current $\overline{j_x}\,a\,b$ passing through the rectangle ab. Adding the contributions from the four sides we have

$$\bar{j}_x ab = c\left[a(M_y)_{z=0} - a(M_y)_{z=b} - b(M_z)_{y=0} + b(M_z)_{y=a}\right].$$

We assume the magnetization $\vec{M}$ to be a continuous function of position, and also that the sides of the rectangle, a and b, respectively are small; therefore we can make a Taylor expansion and obtain the following result:

$$\bar{j}_x a\,b = c\left[-a\cdot\frac{\partial M_y}{\partial z}\cdot b + b\cdot\frac{\partial M_z}{\partial y}\cdot a\right] = c\,a\,b(\nabla\times\vec{M})_x\,.$$

We can find $\bar{j}_y$ and $\bar{j}_z$ in a similar fashion. In general, then

$$\vec{j}_m = c\nabla\times\vec{M} \tag{2.38}$$

That is, the electronic current loops in neutral atoms and molecules give a contribution $\vec{j}_m = c\nabla\times\vec{M}$ to the average current density $\overline{\rho\vec{v}}$.

(2) Method B

We again consider the problem of calculating the average charge density.

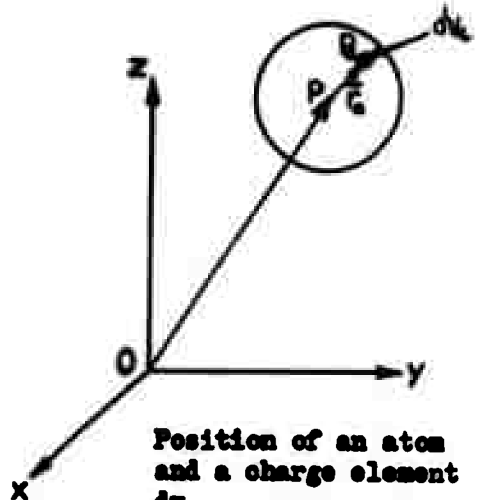
The contribution of free electrons and ions to the average charge density gives rise to a charge density which in Maxwell's theory is called the true charge density $\rho_t$.

In order to calculate the contribution of neutral atoms

and molecules to the average charge density it is convenient to think of the dielectric as broken up into a very large number of "physically infinitesimal" volume elements. We look for the origin of the so-called free charge density of Maxwell's theory in changes of the density of the distribution of atoms and molecules at the boundary of two adjacent volume elements. For purposes of calculation we imagine the whole charge of each atom to be concentrated at some point of the interior of the atom. In this picture there will be no average charge density.

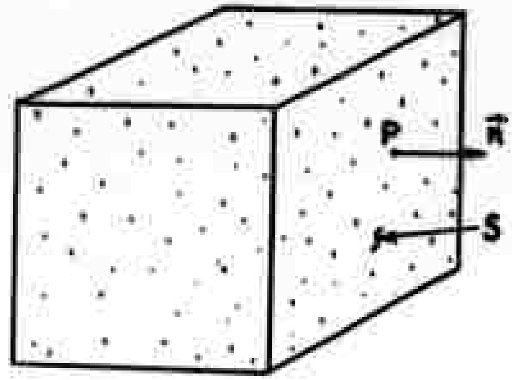
We arrive at the actual charge distribution by moving each charge element back to the point it actually occupies. This will make no difference as far as atoms situated entirely within the boundaries of a "physically infinitesimal" volume element are concerned. However, if an atom is cut by the boundary of this volume element, part of the charge will remain in this volume element, while the rest will contribute to the charge contained in the adjacent volume element. For a uniform distribution of atoms this boundary effect will cancel on the average, but if there is an inhomogeneity in the distribution of atoms a net effect will result. This will be the so-called free charge density. We will regard the elements of charge in a volume element as being in fixed positions and we will also disregard all variations of charge density with time.

We fix the position of an atom by some point P, say the center of the atom. The position of any other point $P_a$ of a given atom is specified by the vector $\vec{r}_a = \vec{PP}_a$. The charge density is assumed to be an explicit function of P and $\vec{r}_a$.



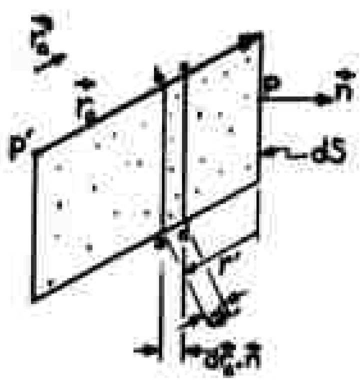

Position of an atom and a charge element $dv_a$.

We group the atoms in a given physically infinitesimal volume element according to their orientation in space. For reasons of simplicity we will consider only atoms of one orientation (that is, atoms which can be made to coincide by a simple translation). To generalize our result for any distribution of atoms we need only to sum over all orientations. We can do this when we define the polarization $\vec{P}$. Let dS be a physically infinitesimal surface element at the point P which separates two adjacent volume elements in which the number of atoms N(P') is assumed to vary from point to point; this atomic density is assumed to be continuous on the physically infinitesimal scale. Consider an atom situated at a point P' in the neighborhood of the boundary dS, and imagine the whole charge of the atom to have been concentrated at P'; let $\rho(P',\vec{r}_a)dv_a$ be an element of charge of this atom which is actually situated at a point $P_a$. If the vector $\vec{r}_a$ crosses the surface dS, the charge $\rho(P',\vec{r}_a)dv_a$, when brought back to its original position,




(a) The "physically infinitesimal" volume element.


(b) Possible positions of atoms in the vicinity of S.




(c) Diagram for evaluation of the charge transported across dS in the direction $\vec{r}_a$.

will be lost to the physically infinitesimal volume element in which the point P' is situated. We add all similar contributions to the charge crossing the surface dS; this is done for all atoms which lie on either side of the surface dS. To perform the summation we first keep the radius vector $\vec{r}_a$ fixed and add up the contributions $\rho(P',\vec{r}_a)dv_a$ from all atoms, then we integrate over the volume of the atom. For a given $\vec{r}_a$ there are equal contributions $\rho(P',\vec{r}_a)dv_a$ from all atoms contained in a cylinder of base dS whose height is the projection of $\vec{r}_a$ on the normal (defined by a unit vector $\vec{n}$) to dS at P. If $\vec{r}_a^{\,o}$ is a unit vector in the direction of $\vec{r}_a$ and if r' is the distance measured from the base in this direction, then the charge transported across dS over the directed-distance $\vec{r}_a$ is given by:

$$\int_0^{r'} N(P')\, dSdr'\, (\vec{r}_a^{\,o}\cdot\vec{n})\cdot\rho(P',r_a)\, dv_a\,. \qquad (2.39)$$

The slowly varying functions $\rho(P',\vec{r}_a)$ and N(P') may be expanded about the point P giving

$$\rho(P',\vec{r}_a) = \rho(P+P'-P,\vec{r}_a) \approx \rho(P,\vec{r}_a) + \overrightarrow{(P'-P)}\cdot\nabla_P\, P + \ldots ,$$

$$N(P') = N(P+P'-P) \approx N(P) + \overrightarrow{(P'-P)}\cdot\nabla_P\, N + \ldots . \qquad (2.40)$$

$$\therefore\quad \rho(P',\vec{r}_a)\, N(P') \approx \rho(P,\vec{r}_a)\, N(P) - r'\vec{r}_a^{\,o}\cdot\nabla_P\, \rho N + \ldots$$

We keep only the first term of the expansion and on substituting this term into the equation (2.39) we get:

$$\sigma_f \, dS = N(P)\,\rho\,(P,\vec{r}_a)\,(\vec{r}_a^{\,\circ} \cdot \vec{n})\,dS\,dv_a \int_0^{r'} dr' \, . \tag{2.41}$$

We obtain the net charge which crosses the surface element dS, in the direction $\vec{n}$ (when the charge distribution in the atoms is restored) by integrating over the volume of the atom. This gives:

$$\sigma_f \, dS = \vec{n} \cdot \int_{\text{atom}} \vec{r}_a\,\rho\,(P,\vec{r}_a)\,N(P)\,dv_a \, , \tag{2.42}$$

since $r'\vec{r}_a^{\,\circ} = \vec{r}_a$.

To calculate the free charge $\rho_f\,dv$ contained in a physically infinitesimal volume element we integrate the expression (2.42) over the entire boundary of the volume element, the transport of charge at each point is taken in the inward direction. The space density of free charge is thus defined by

$$\rho_f \, dv = -\int_{\text{boundary of } dv} \sigma_f \, dS \, . \tag{2.43}$$

We define the electric dipole of the atom at the point P by

$$p(P) = \int_{\text{atom}} \vec{r}_a\,\rho\,(P,\vec{r}_a)\,dv_a \, , \tag{2.44}$$

and then rewrite the expression (2.42) in the following form:

$$\sigma_f \, dS = \vec{p}N \cdot \vec{n}\,dS \, . \tag{2.45}$$

The dipole moment per unit volume or the polarisation $\vec{P}$ (this is a macroscopic quantity) can be written as

$$\vec{P}(P) = N(P)\,\vec{p}(P) \,. \tag{2.46}$$

If we want to generalise our results to include atoms of different spatial orientations we sum over all orientations and write the polarisation as

$$\vec{P}(P) = \sum_{\emptyset} N_{\emptyset}(P)\,\vec{p}_{\emptyset}(P) \,. \tag{2.47}$$

We can rewrite the equation in terms of the polarisation and obtain the result

$$\sigma_f \, dS = \vec{P} \cdot \vec{n} \, dS \,. \tag{2.48}$$

If we insert this result in equation (2.43) we obtain

$$\rho_f \, dv = - \int_{\text{boundary of } dv} \vec{P} \cdot \vec{n} \, dS \quad , \tag{2.49}$$

and on making use of Gauss' theorem, we find

$$\rho_f \, dv = - \nabla \cdot \vec{P} \, dv \quad , \quad \text{i.e.} \quad \rho_f = - \nabla \cdot \vec{P} \,. \tag{2.50}$$

We can now write the average charge density as follows

$$\bar{\rho} = \rho_t + \rho_f = \rho_t - \nabla \cdot \vec{P} \,. \tag{2.51}$$

If we substitute this result in equation (2.2b) we get

$$\nabla \cdot \vec{E} = 4\pi \bar{\rho} = 4\pi \rho_t - \nabla \cdot 4\pi \vec{P} ,$$

$$\nabla \cdot (\vec{E} + 4\pi \vec{P}) = 4\pi \rho_t . \qquad (2.52)$$

We can write this equation in the Maxwellian form (see equation (2.1b)) by introducing the electric displacement $\vec{D} = \vec{E} + 4\pi \vec{P}$, and hence we have:

$$\nabla \cdot \vec{D} = 4\pi \rho_t . \qquad (2.53)$$

We now proceed to outline the evaluation of the average current density. With the exception of certain characteristic differences the calculation of the average current density is analagous to that of the average charge density.

In order to evaluate the average current density the charge density is assumed to be a function of the time t, as well as of the position P of the atom and the radius vector $\vec{r}_a$. However, only slow variations of charge and current densities are taken into account. Any rapid fluctuations are smoothed out by averaging. Since such time-averages do not change the form of the expressions to which they are applied, they are left out from the following considerations. In addition, it is customary to neglect the time-variations of the distribution of atoms and to consider only stationary distributions of atoms characterised by a spatial function N(P').

As in method A the motion of ions and free electrons give rise to a conduction current density $\vec{j}_c$.

Neutral atoms contribute to the average current density even when their distribution is uniform since the integral

$$\int_{\text{atom}} \{\rho\vec{v}\}\,(P,t,\vec{r}_a)\,dv_a \qquad (2.54)$$

does not in general vanish. The contribution of neutral atoms to the average current density is called the polarisation current density and is given by the expression

$$\vec{j}_p = N(P)\int_{\text{atom}} \{\rho\vec{v}\}\,(P,t,\vec{r}_a)\,dv_a \ . \qquad (2.55)$$

The spatial inhomogeneity of the distribution of atoms gives rise to the magnetisation current density,

$$\vec{j}_m\,dv = -\int_{\text{boundary of } dv} dS\cdot N(P)\int_{\text{atom}} \{\rho\vec{v}\}\,(P,t,\vec{r}_a)\,(\vec{n}\cdot\vec{r}_a)\,dv_a \ .$$

A certain simplification in notation results if we make use of the relationship

$$\vec{j}(P,t,\vec{r}_a) = \{\rho\vec{v}\}\,(P,t,\vec{r}_a) \ . \qquad (2.56)$$

With this notation we can write

$$\vec{j}_p = N(P)\int_{\text{atom}} \vec{j}\,(P,t,\vec{r}_a)\,dv_a \ , \qquad (2.57)$$

$$\vec{j}_m\,dv = -\int_{\text{boundary of } dv} dS\cdot N(P)\int_{\text{atom}} \vec{j}(P,t,\vec{r}_a)\,(\vec{n}\cdot\vec{r}_a)\,dv_a \ . \qquad (2.58)$$

In all of the above expressions we have limited the Taylor expansion of the atomic distribution $N(P')$ to its first term, that is, to $N(P)$ and similarly we have limited the expansion of $\vec{j}(P',t,\vec{r}_a)$ to its first term $\vec{j}(P,t,\vec{r}_a)$. This is consistent with the expansions used in evaluating the average charge density.

The average current density can therefore be written as the sum of three terms

$$\overline{\rho \vec{v}} = \vec{j}_e + \vec{j}_p + \vec{j}_m \, , \qquad (2.59)$$

where $\vec{j}_e$, $\vec{j}_p$ and $\vec{j}_m$ have been defined above.

We now proceed to transform the expressions for $\vec{j}_p$ and $\vec{j}_m$ into expressions which are more familiar.

We multiply the equation of continuity $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$ by $\vec{r}_a$ and then integrate the resulting expression over the volume of the atom. We obtain the relation

$$\int_{\text{atom}} \vec{r}_a \frac{\partial \rho}{\partial t} \, dv_a = - \int_{\text{atom}} \vec{r}_a \, \nabla \cdot \vec{j} \, dv_a \, . \qquad (2.60)$$

(the divergence is taken with respect to the coordinates of the point $P_a$ while the point $P$ is kept fixed.)

The right hand side may be simplified by performing a partial integration. Of the two resulting terms only one remains, since

in one we are left with a surface integral which can be made to vanish. Equation (2.60) becomes

$$\int_{\text{atom}} \vec{r}_a \frac{\partial \rho}{\partial t}\, dv_a = \int_{\text{atom}} \vec{j}\, dv_a . \tag{2.61}$$

From the definition of the electric dipole moment

$$\vec{p} = \int_{\text{atom}} \vec{r}_a\, \rho\, dv_a ,$$

we obtain, by differentiating with respect to t, the relation

$$\frac{\partial \vec{p}}{\partial t} = \int_{\text{atom}} \vec{r}_a \frac{\partial \rho}{\partial t}\, dv_a .$$

We have therefore

$$\frac{\partial \vec{p}}{\partial t} = \int_{\text{atom}} \vec{j}\, dv_a . \tag{2.62}$$

We can write the expression for the polarisation current density as follows:

$$\begin{aligned}\vec{j}_p &= N(P) \int_{\text{atom}} \vec{j}(P,t,\vec{r}_a)\, dv_a ,\\ &= N(P) \frac{\partial \vec{p}}{\partial t} = \frac{\partial}{\partial t} N(P)\vec{p} ,\\ &= \frac{\partial}{\partial t} \vec{P} ,\end{aligned} \tag{2.63}$$

where we have defined $\vec{P} = N\vec{p}$.

We can simplify the expression for the magnetisation current density $\vec{j}_m$ by making use of the relation

$$\int_{\text{atom}} (\vec{n}\cdot\vec{j})\vec{r}_a \, dv_a + \int_{\text{atom}} (\vec{n}\cdot\vec{r}_a)\vec{j} \, dv_a = 0 \qquad (2.64)$$

which follows from the equations (2.23) - (2.25). The equation (2.64) can be written in the form

$$-\int_{\text{atom}} (\vec{n}\cdot\vec{r}_a)\vec{j} \, dv_a = \frac{1}{2}\int_{\text{atom}} [(\vec{n}\cdot\vec{j})\cdot\vec{r}_a - (\vec{n}\cdot\vec{r}_a)\vec{j}] \, dv_a$$

$$= c\vec{n} \times \int_{\text{atom}} \frac{\vec{r}_a \times \vec{j}}{2c} \, dv_a$$

$$= c\vec{n} \times \vec{m} \, . \qquad (2.65)$$

where:

$$\vec{m} = \frac{1}{2c}\int_{\text{atom}} (\vec{r}_a \times \vec{j}) \, dv_a \, . \qquad (2.66)$$

We can now write the expression for $\vec{j}_m$ as follows:

$$\vec{j}_m\, dv = -\int_{\text{boundary of } dv} dS \cdot N(P) \cdot \int_{\text{atom}} \vec{j}(\vec{n} \cdot \vec{r}_a)\, dv_a$$

$$= \int_{\text{boundary of } dv} dS\, N(P)\, c\, \vec{n} \times \vec{m}$$

$$= +\, c \int_{\text{boundary}} \vec{n} \times \vec{M}\, dS \qquad (\text{since } \vec{M} = N\vec{m})$$

$$= +\, c\nabla \times \vec{M} dv\ . \tag{2.67}$$

Therefore we have $\vec{j}_m = +\, c\nabla \times \vec{M}$ as desired.

## 2.4 Rytov's Modification of the Electromagnetic Equations

By considering the derivation of Maxwell's equations from the 'microscopic equations' we have attained a more favourable vantage-point from which we can follow Rytov's modification of electromagnetic equations. The 'modified equations' provide us with a general theory of electromagnetic fluctuation phenomena.

In evaluating the average charge and current densities we have assumed:

(a) the charge elements to be in fixed positions,

(b) the distribution density of atoms to be a function of position only, and

(c) the current density to be a slowly smoothly varying function of the time; any rapid variations in the current density having been smoothed out.

With the aid of these assumptions we have obtained the following expressions for the average charge and current densities:

$$\bar{\rho} = -\nabla \cdot \vec{P} \qquad (2.68)$$

$$\overrightarrow{\rho v} = \dot{\vec{P}} + c\nabla \times \vec{M} \, . \qquad (2.69)$$

(In the above expressions we have omitted the contributions arising from the ions and free electrons.)

To obtain equation (1.3) of Rytov's monograph[1] we use the following relationships:

$$\vec{E} = \bar{\vec{e}} \quad , \quad \vec{B} = \bar{\vec{h}} \quad , \quad \vec{B} = \mu\vec{H} = \vec{H} + 4\pi\vec{M} \quad , \quad \vec{D} = \varepsilon\vec{E} = \vec{E} + 4\pi\vec{P} \qquad (2.70)$$

to obtain the expressions

$$\vec{P} = \frac{(\varepsilon - 1)}{4\pi} \bar{\vec{e}} \; , \qquad (2.71)$$

$$\vec{M} = \frac{(\mu - 1)}{4\pi\mu} \bar{\vec{h}} \; . \qquad (2.72)$$

If, in addition, we assume that the electric field intensity has a time-dependence of the form $e^{i\omega t}$, we can rewrite equations (2.68) and (2.69) in the form:

$$\bar{\rho} = -\nabla \cdot \left[ \frac{(\varepsilon - 1)}{4\pi} \bar{\vec{e}} \right] , \qquad (2.73)$$

$$\rho\bar{\vec{v}} = \frac{(\varepsilon-1)\,i\omega}{4\pi}\,\bar{\vec{e}} + \frac{c}{4\pi}\,\nabla\times\left[\frac{(\mu-1)}{\mu}\,\bar{\vec{h}}\right]. \qquad (2.74)$$

__Note:__ Our notation, which is the customary one, differs from Rytov's in that we use small letters to denote microscopic quantities while he uses capital letters. We denote the macroscopic quantities by capital letters while Rytov uses capital letters with a bar written over them. Since we have used the letter M to denote the magnetic polarization we will use the letter L in place of Rytov's M. For the time being we will use our own notation.

Contrary to our assumptions, in any body in thermal equilibrium at a temperature T both the position and the motion of the charge constituents as well as the density distribution of the atoms are undergoing random oscillations due to thermal agitation.* Without going into any detailed quantitative derivation Rytov assumes that these random oscillations of the charges and currents can be accounted for (or described by) the addition of two terms in the expressions for the average charge and current densities. In particular, Rytov takes the microcharge and the microcurrent densities to be:

$$\rho = -\nabla\cdot\left[\frac{(\varepsilon-1)}{4\pi}\,\vec{e} + \frac{(\varepsilon-1)}{4\pi}\,\vec{k}\right]. \qquad \begin{matrix}(2.75)\\(R,1.7)^{**}\end{matrix}$$

* We have been interested only in mean linear values and hence the above results do not take into account any fluctuations.

** By (R,1.7) we mean eqn.(1.7) of Rytov's monograph.

$$\vec{\rho v} = \frac{(\varepsilon-1)i\omega}{4\pi}\vec{e} + \frac{(\varepsilon-1)i\omega}{4\pi}\vec{k} + c\nabla\times\left[\frac{\mu-1}{4\pi\mu}(\vec{h}+\vec{l})\right] \qquad (2.76)$$
(R,1.6)

That is, in addition to the fields $\vec{e}$ and $\vec{h}$ Rytov introduces a random electric field $\vec{k}$ and a random magnetic field $\vec{l}$; all four fields act on the microcharges and the microcurrents.

We can think of the fields $\vec{k}$ and $\vec{l}$ as producing the random or oscillating motion which is superimposed on the average motion of the charges and currents.

Another equivalent interpretation is also possible. If we consider the expressions for the average charge and current density and make use of the relations

$$i\omega\frac{(\varepsilon-1)}{4\pi}\overline{\vec{e}} = \dot{\vec{P}} \qquad (2.78)$$

$$\frac{(\mu-1)}{4\pi\mu}\overline{\vec{h}} = \vec{M} \qquad (2.79)$$

we will obtain the following expressions for the average charge and current densities:

$$\bar{\rho} = -\nabla\cdot\vec{P} - \nabla\frac{(\varepsilon-1)}{4\pi}\vec{k} \qquad (2.80)$$

$$\overline{\rho\vec{v}} = \dot{\vec{P}} + \frac{(\varepsilon-1)i\omega}{4\pi}\overline{\vec{k}} + c\nabla\times\vec{M} + c\nabla\times\left[\frac{(\mu-1)}{4\pi\mu}\overline{\vec{l}}\right]. \qquad (2.81)$$

By analogy with equations (2.78) and (2.79) we define the relations:

$$\vec{P}_f = \frac{(\varepsilon-1)\,i\,\omega}{4\pi}\,\overline{\vec{k}}\;, \qquad (2.82)$$

$$\vec{M}_f = \frac{\mu-1}{4\pi\,\mu}\,\overline{\vec{l}}\;, \qquad (2.83)$$

and $\therefore$ write

$$\vec{\rho} = -\,\nabla\cdot(\vec{P}+\vec{P}_f)\;, \qquad (2.84)$$

$$\overline{\rho\vec{v}} = (\dot{\vec{P}}+\dot{\vec{P}}_f) + c\nabla\times(\vec{M}+\vec{M}_f)\;. \qquad (2.85)$$

Now $\vec{P}$ and $\vec{M}$ represent the electric and magnetic dipole moments per unit volume, respectively. We can therefore regard $\vec{P}_f$ and $\vec{M}_f$ as spontaneous local electric and magnetic dipole moments per unit volume. $\vec{P}_f$ and $\vec{M}_f$ are due to the random thermal motion of the charge and currents.

The modified microscopic equations now take the form

$$\nabla\times\vec{e} = -\,i\,k\,\vec{h}\;,$$

$$\nabla\times\frac{\vec{h}}{\mu} = i\,k\,\varepsilon\,\vec{e} + ik(\varepsilon-1)\vec{k} + \nabla\times\frac{\mu-1}{\mu}\,\vec{l}\;, \qquad \begin{array}{c}(2.86)\\(R,1.8)\end{array}$$

$$\nabla\cdot\varepsilon\vec{e} = -\,\nabla\cdot(\varepsilon-1)\dot{\vec{k}}\;,$$

$$\nabla\cdot\vec{h} = 0\;,$$

If we let $\quad \vec{h}' = \vec{h} - \frac{(\mu-1)}{\mu}(\vec{h}+\vec{l}) = \frac{\vec{h}}{\mu} - \frac{\mu-1}{\mu}\,\vec{l}$ , then the first two equations can be written in the symmetrical form:

$$\nabla \times \vec{e} = -ik\mu \vec{h}' - ik(\mu - 1)\,\vec{l}\,, \qquad (2.87)$$

$$\nabla \times \vec{h}' = ik\varepsilon\vec{e} + ik(\varepsilon - 1)\,\vec{k}\,. \qquad (R,1.9)$$

If we perform a space-time average over a "physically infinitesimal" volume the above equations take the form:

$$\nabla \times \vec{E} = -ik\mu \vec{H}' - ik4\pi\mu \vec{M}_f = -ik\mu(\vec{H}' + 4\pi \vec{M}_f), \qquad (2.88)$$

$$\nabla \times \vec{H}' = ik\varepsilon \vec{E} + ik4\pi \vec{P}_f = ik(\varepsilon\vec{E} + 4\pi \vec{P}_f)\,,$$

where we have made use of eqns. (2.82) and (2.83) and have defined $\bar{\vec{h}}' = \vec{H}'$.

When the equations are written in this form it is natural to regard the terms on the right hand side of the equations which involve $\vec{M}_f$, $\vec{P}_f$ as distributed sources which give rise to the fields $\vec{E}$ and $\vec{H}'$. The fluctuations of E and H' and the fluctuations of all quantities depending on $\vec{E}$ and $\vec{H}'$ are evaluated in terms of $\vec{P}_f$ and $\vec{M}_f$.

Equation (2.87), when written in Rytov's notation, become:

$$\nabla \times \vec{E} = -ik\mu \vec{H}' - ik(\mu - 1)\vec{M}\,, \qquad (2.89)$$

$$\nabla \times \vec{H}' = ik\varepsilon\vec{E} + ik(\varepsilon - 1)\vec{K}\,. \qquad (R,1.9)$$

If the magnetic losses are zero, equation (2.89) reduces to:

$$\nabla \times \vec{E} = -ik\mu\vec{H} \ , \tag{2.90}$$

$$\nabla \times \vec{H} = ik\varepsilon\vec{E} + ik(\varepsilon - 1)\vec{K} \ . \tag{E,1.10}$$

(By $\vec{H}$ we understand here, and in what is to follow, the vector $H'$, i.e.

$$\vec{H}' = \vec{H} - \frac{(\mu - 1)}{\mu}(\vec{H} + \vec{M}) = \frac{\vec{H}}{\mu} - \frac{\mu - 1}{\mu}\vec{M} \ .$$

The problem of finding the electromagnetic fluctuations in any medium has been reduced to solving a boundary value problem. We look for solutions of equations (2.89) or if there are no magnetic losses in the medium, for solutions of (2.90), subject to the boundary conditions that the tangential components of $\vec{E}$ and $\vec{H}$ are continuous at the boundary of the medium, i.e.

$$\begin{aligned} \vec{n} \times (\vec{E}_1 - \vec{E}_2) &= 0 \ , \\ \vec{n} \times (\vec{H}_1 - \vec{H}_2) &= 0 \ , \end{aligned} \tag{2.91}$$

where: $\vec{E}_1, \vec{H}_1$ are the fields on one side of the boundary (region 1) and

$\vec{E}_2, \vec{H}_2$ the fields in the other region (2).

$\vec{n}$ is a unit vector normal to the boundary and pointing from region 1 to region 2.

We are to regard the quantities $\vec{K}$ and $\vec{H}$ as known functions of position.

Consider the solutions of equations (2.90). We eliminate $\vec{H}$ from the first equation by taking the curl of the first equation and then making use of the second equation. In this way we get the vector wave equation

$$\nabla \times \nabla \times \vec{E} - k^2 \mu \varepsilon \vec{E} = k^2 \mu (\varepsilon - 1) \vec{K} ,$$

$$\text{grad div } \vec{E} - \nabla^2 \vec{E} - k^2 \mu \varepsilon \vec{E} = k^2 \mu (\varepsilon - 1) \vec{K} . \qquad (2.92)$$

(We have assumed $\varepsilon$ and $\mu$ to be independent of position, that is we are considering a homogeneous, isotropic medium.) Because the equation is linear we can write the solution in the following form:

$$E_i(\vec{r}) = \int_V G^E_{ik} (\vec{r}, r') \, K_k(\vec{r}') \, dr' . \qquad (2.93)$$

We proceed in a similar fashion to find

$$\vec{H}_j(\vec{r}) = \int_V G^H_{jl} (\vec{r}, \vec{r}'') \, K_l(\vec{r}'') \, d\vec{r}'' , \qquad (2.94)$$

(where the G's are known functions; a repeated index implies that a summation is to be carried out).

In general we are interested in the mean values of quantities which are bilinear in the components of $\vec{E}$ and $\vec{H}$. One of

the most important bilinear expressions is the space-time average of the Poynting vector, i.e.:

$$\vec{S}_\omega = \frac{c}{4\pi}\left[\overline{\vec{E} \times \vec{H}^*} + \overline{\vec{E}^* \times \vec{H}}\right] . \tag{2.95}$$

The expression without the bar is the time-average of the Poynting vector and the bar over the terms indicates that a space-average is to be performed. This expression is derived in the next section. Hence any component of (2.95) will involve terms of the form

$$\frac{c}{4\pi}\int G^{E}_{i\alpha}(\vec{r},\vec{r}\,')\, G^{H}_{j\beta}(\vec{r},\vec{r}\,'')\, \overline{K_\alpha(\vec{r}\,')\, K_\beta(\vec{r}\,'')}\, d\vec{r}\,'\, d\vec{r}\,'' . \tag{2.96}$$

In order to evaluate this expression we will need to know the quantity $\overline{K_\alpha(\vec{r}\,')\, K_\beta(\vec{r}\,'')}$ , that is, the space average of the products of the components of $\vec{K}$ at various points. The quantity $\overline{K_\alpha(\vec{r}\,')\, K_\beta(\vec{r}\,'')}$ is called the space correlation function and will be denoted by

$$F_{\alpha\beta}(\vec{r}\,',\vec{r}\,'') = \overline{K_\alpha(\vec{r}\,')\, K_\beta(\vec{r}\,'')} , \quad (\alpha,\beta=1,2,3) . \tag{2.97}$$

We now proceed to derive equation (2.95) and also to outline Rytov's arguments concerning the explicit functional dependence of $F_{\alpha\beta}(\vec{r}\,',\vec{r}\,'')$.

## III STATISTICAL FORMULATION

The solution of the electrodynamic equations derived in the previous section will lead to expressions for the instantaneous and localised electro-magnetic fields in the medium. However, as pointed out earlier, the intensity and density of radiation in this phenomenological theory are defined in terms of average intensity fluctuations of these microscopic fields. The statistical formulation of this problem can thus be resolved into two main subjects: the time-average and the space-average of the intensity fluctuations.

### 3-1. Time-Average Value of the Poynting Vector

The instantaneous value of the electric and magnetic fields E(t) and H(t) may be expressed as Fourier series expansions (considering positive frequencies)

$$\vec{E}(t) = \sum_{n=0}^{+\infty} \vec{E}_n \, e^{i\omega_n t} \, ,$$

$$\vec{H}(t) = \sum_{n=0}^{+\infty} \vec{H}_n \, e^{i\omega_n t} \, , \tag{3.1}$$

with $$\omega_n = \frac{2\pi n}{T} \, .$$

T is a time interval greater than any correlation time in the uniformly random processes under study.

The time-average of the vector product $\vec{E}(t) \times \vec{H}(t)$ leads to:

$$\overline{\vec{E}(t) \times \vec{H}(t)}^{t} = \frac{1}{T} \int_{-T/2}^{+T/2} \vec{E}(t) \times \vec{H}(t)\, dt \,,$$

and from Eqn.(3.1)

$$\overline{\vec{E}(t) \times \vec{H}(t)}^{t} = \sum_{n,m}^{+\infty} \vec{E}_n \times \vec{H}_m^{*}\, \delta_{nm} = \sum_{n=\infty}^{+\infty} (\vec{E}_n \times \vec{H}_n^{*} + \vec{E}_n^{*} \times \vec{H}_n)$$

$$= [E(t) \times H^{*}(t) + E^{*}(t) \times H(t)] \,. \qquad (3.2)$$

to facilitate the solution of the basic electro-magnetic equations (2.89) it is desirable to express the instantaneous fields $E(t)$ and $H(t)$ by Fourier integrals. This can be done if

$$\lim_{T\to\infty} \left( \frac{TE_n}{2\pi} \right) = \vec{E}(\omega) \,, \qquad (3.3)$$

and

$$\lim_{T\to\infty} \left( \frac{TH_m}{2\pi} \right) = \vec{H}(\omega) \,.$$

i.e.

$$E_n \sim H_n \sim \frac{1}{T} \text{ for large } n \,.$$

Then we have

$$\vec{E}(t) = \int_{-\infty}^{+\infty} \vec{E}(\omega)\, e^{i\omega t}\, d\omega \,,$$

$$\vec{H}(t) = \int_{-\infty}^{+\infty} \vec{H}(\omega)\, e^{i\omega t}\, d\omega \,. \qquad (3.4)$$

However, due to the condition (3.3), the limit of the time-average product $\overline{\vec{E}(t) \times \vec{H}(t)}^{t}$ as $T \to \infty$ is zero. i.e.

$$\lim_{T\to\infty} \overline{\vec{E}(t) \times \vec{H}(t)}^{t} = \lim_{T\to\infty} \frac{\pi}{T} \int_{-\infty}^{+\infty} [\vec{E}(\omega) \times \vec{H}^{*}(\omega) + \vec{E}^{*}(\omega) \times \vec{H}(\omega)]\, d\omega = 0 .$$

Rytov gets around this difficulty by defining a reduced spectral distribution $G(\omega)$ such that

$$\lim_{T\to\infty} \frac{T}{4\pi} \left( E_n \times H_n^{*} + E_n^{*} \times H_n \right) = G(\omega) , \qquad (3.5)$$

where $G(\omega)$ is finite.

It follows that:

$$\lim_{T\to\infty} \overline{E(t) \times H(t)}^{t} = \int_{-\infty}^{+\infty} G(\omega)\, d\omega .$$

The integral expansion given by (3.4) is then applied, for T large but finite, by assuming that

$$E(\omega) \times H^{*}(\omega) = \tilde{E}(\omega) \times \tilde{H}^{*}(\omega) ,$$

and

$$E_n \times H_n^{*} = \tilde{E}_n \times \tilde{H}_n^{*} \qquad (3.6)$$

where the sign ~ represents the values of E and H as T becomes large but finite.

The time-average Poynting vector can thus be written, according to Eqn.(3.2), as

$$\bar{S}^t = \frac{c}{4\pi}\left[\overline{E(t) \times H(t)}^t\right] = \frac{c}{4\pi}\left[E(t) \times H^*(t) + E^*(t) \times H(t)\right] , \quad (3.7)$$

where the formal integral expansion (3.4) is used as a useful notation for the series (3.1) and the spectral intensities are defined according to Eqns.(3.5) and (3.6).

3-2. Space-average of the Intensity of the Fluctuations

The solution of the basic differential equations (2.89), satisfying the boundary conditions (2.91) leads to a representation of the E and H fields in terms of a volume integral involving the various components of the lateral field K. The energy of radiation is dependent not only on the time-average but also on the space-average of the product of the components of E and H fields (Poynting vector) and therefore will involve space-average of the product of the various components of the lateral field K at various points in the system. In terms of statistical theories this average of the product of the random field K at various points in the medium. is defined by the space correlation function $F_{\alpha\beta}(\vec{r}',\vec{r}'')$.

$$F_{\alpha\beta}(r',r'') = \overline{K_\alpha'(r')\, K_\beta''(r'')} , \qquad (3.8)$$

where $\alpha, \beta = 1,2,3$ (the three orthogonal components of $\vec{K}$).

This function contains the "space correlation coefficient" plus the magnitude of the thermal fluctuations at point r', r''. Considering an homogeneous and isotropic medium, the correlation

functions $F_{\alpha\beta}(r', r'')$ form a tensor and their values depend solely on $\vec{r} = |\vec{r}' - \vec{r}''|$. The general expression for the tensor dependence of r only is

$$F_{\alpha\beta}(r) = f(r)\delta_{\alpha\beta} + g(r)\,\frac{x_\alpha x_\beta}{r^2}\,, \qquad (3.9)$$

where $\delta_{\alpha\beta}$ is the unit tensor and $x_{\alpha\beta}$ is the projection of r along the three orthogonal axes ($\alpha, \beta = 1,2,3$). Eqn.(3.9) can also be written in the following way

$$F_{\alpha\beta}(r) = \phi(r)\delta_{\alpha\beta} + \frac{\partial^2\psi(r)}{\partial x_\alpha \partial x_\beta}\,, \qquad (3.10)$$

where

$$f(r) = \phi(r) + \frac{1}{r}\frac{d\psi(r)}{dr} \quad ; \quad g(r) = \frac{d^2\psi(r)}{dr^2} - \frac{1}{r}\frac{d\psi(r)}{dr}\,.$$

In applying this correlation function to the problem of thermal radiation Rytov makes the two following assumptions:

$$\phi(r) = C\,\delta(r) = C\,\delta(x_1)\,\delta(x_2)\,\delta(x_3)\,, \qquad (3.11)$$

$$\psi(r) = 0\,,$$

so that

$$F_{\alpha\beta}(r) = C\,\delta_{\alpha\beta}\,\delta(\vec{r})\,. \qquad (3.12)$$

where: C is a constant and $\delta(r)$ is the Dirac delta function.

This expression for the correlation function $F_{\alpha\beta}(r)$ is then justified in the following way.

In a bounded medium the assumption that $F_{\alpha\beta}$ depends only on $|r' - r''|$ is not valid for points lying within the correlation radius from the surface. The present theory assumes that the microscopic inhomogeneities (atoms, molecules) of the medium are very small in comparison with the macroscopic inhomogeneities. Since $\vec{K}$ includes fields from any source, in particular fields due to the thermal motion of the micro-charges (sources of thermal radiation) it is reasonable to assume that the correlation radius is of the order of magnitude of the microcharges. It is even permissible to admit a zero correlation radius $[\phi(r) = C\delta(\vec{r})]$ because the wavelengths of the emitted radiation and the dimensions of the bodies are very large compared with the correlation radius. Under this condition, the dependence on $|r' - r''|$ of $F_{\alpha\beta}$ is valid up to the surface of the bounded medium.

The choice of the functions $\phi(r)$ and $\psi(r)$ given in Eqn.(3.11) does not involve singularities when solving the problem of radiation into a transparent medium. In fact it leads to the expected results in the limiting cases for which solutions have been obtained from other theories (Clausius, Nyquist). However, when studying a body radiating into an absorbing medium the use of the conditions (3.11) leads to a divergence of the expressions for the intensity ($I_\omega$) and the electrical ($U_{e\omega}$) and magnetic ($U_{m\omega}$) densities of energy at regions near the boundaries. It is found that the expressions for $I_\omega$ and $U_{m\omega}$ are independent of $\psi(r)$ and

therefore the choice $\phi(r) = 0$ may be justified for these quantities. However, the expression for $U_{e\omega}$ depends on $\phi(r)$. To make $U_{e\omega}$ convergent it is necessary to assume $\phi(r) = \frac{C}{r}$, but this still leaves $I_\omega$ and $U_{m\omega}$ divergent.

This difficulty may be removed if the correlation radius is assumed different from zero, that is by writing

$$\phi(r) = C\,\frac{e^{-r^2/a^2}}{(\pi a^2)^{3/2}}\,, \tag{3.13}$$

where a is the effective correlation radius.

It is then found[1] that the condition $\phi(r) = 0$ may be applied and that all three quantities $I_\omega$, $U_{e\omega}$ and $U_{m\omega}$ become convergent. In this way the choice of $\phi(r) = 0$ is justified for both transparent and absorbing media. The zero-correlation radius may be applied for transparent medium and a finite correlation radius of the form (3.13) is necessary for radiation in an absorbing medium.

Since Rytov's[(1)] arguments concerning the correlation function $F_{\alpha\beta}(\vec{r}\,',\vec{r}\,'')$ are somewhat heuristic in nature we shall outline Landau and Lifschits' treatment[(6)] which is more general and more mathematically satisfying. They base their treatment on a method developed by Callen and Welton[(7)]. The fluctuations of physical quantities belonging to a physical system acted upon by an external disturbance are related to the dissipative properties of the system.

### 3.3 The Correlation Function in Terms of the Dissipative Properties of the System

Consider a system in thermodynamic equilibrium and let $x(t)$ be any time dependent physical quantity describing the state of the system, and assume that $x$ is undergoing fluctuations about its mean value $\bar{x}$ (for simplicity let $\bar{x} = 0$).

For a stationary state the correlation function $\overline{x(t)x(t')}$ may be written in the form

$$\overline{x(t)x(t')} = \phi(t'-t) \; . \tag{3.14}$$

The bar indicates an averaging over the probabilities of all values which the quantity $x(t)$ can take at the given times $t$ and $t'$; such statistical averaging is equivalent to a time average over the values of the time $t$ for a given value of the time interval $t'-t$. It is convenient to work with the Fourier transform of $\phi(t'-t)$. If we let

$$x(t) = \int_{-\infty}^{+\infty} x_\omega \, e^{-i\omega t} \, dt \; , \tag{3.15}$$

where

$$x_\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} x e^{+i\omega t} \, dt \; , \tag{3.16}$$

we obtain the relation

$$\phi(t'-t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \overline{x_\omega \, x_{\omega'}} \, e^{i(\omega t + \omega' t')} \, d\omega d\omega' \; .$$

For the right-hand side to be a function of $(t' - t) = \tau$ only, $\overline{x_\omega x_{\omega'}}$ must be of the form

$$\overline{x_\omega x_{\omega'}} = (x^2)_\omega \, \delta(\omega + \omega') \ . \tag{3.17}$$

where $\delta(\omega + \omega')$ is the Dirac Delta function.
If we substitute equation (3.17) into the expression for $\phi(t' - t)$ we obtain the result

$$\phi(t' - t) = \phi(\tau) = \int_{-\infty}^{+\infty} (x^2)_\omega \, e^{-i\omega\tau} \, d\omega \ . \tag{3.18}$$

$\phi(0)$ is just the mean square of the fluctuating quantity itself, for we have

$$\overline{x^2} = \int_{-\infty}^{+\infty} (x^2)_\omega \, d\omega \tag{3.19}$$

The "spectral density" of the mean square fluctuations is just the quantity $(x^2)_\omega$. From equation (3.18) we find

$$(x^2)_\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(\tau) \, e^{i\omega\tau} \, d\tau \ , \tag{3.20}$$

that is, $(x^2)_\omega$ is also the Fourier transform of the correlation function.

If we have several fluctuating quantities $x_1$, $x_2$ ... $x_n$ describing the state of the system, which simultaneously fluctuate about their equilibrium values we define the correlation functions $\phi_{ik}$ by

$$\phi_{ik}(t - t') = \overline{x_k(t) \, x_i(t')} \ , \tag{3.21}$$

$$(i,k = 1, \ldots n)$$

and if we let $\tau = t' - t$ this becomes

$$\phi_{ik}(\tau) = \overline{x_i(t+\tau)\, x_k(t)} \quad . \tag{3.22}$$

The correlation functions satisfy the relations

$$\phi_{ik}(\tau) = \phi_{ki}(-\tau) \quad . \tag{3.23}$$

In place of equation (3.20) we now have

$$(x_i\ x_k)_\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi_{ik}\ e^{i\omega\tau}\ d\tau \quad . \tag{3.24}$$

If we wish our results to be valid for quantum-mechanical quantities we must replace the classical quantity x by its quantum mechanical operator $\hat{x}(t)$. In this case we must symmetrise the correlation function and write

$$\phi(t' - t) = \frac{1}{2} [\ \overline{\hat{x}(t)\ \hat{x}(t') + \hat{x}(t')\ \hat{x}(t)}\ ] \quad . \tag{3.25}$$

This is necessary since, in general, the operators $\hat{x}(t)$ and $\hat{x}(t')$ do not commute for different instants of time. The bar indicates an averaging by means of the Gibb's distribution function. The relation corresponding to that given by equation (3.17) is

$$\overline{\frac{1}{2}\,(\hat{x}_\omega\ \hat{x}_{\omega'} + \hat{x}_{\omega'}\ \hat{x}_\omega)} = (x^2)_\omega\ \delta(\omega + \omega') \quad . \tag{3.26}$$

To calculate the mean square value of the fluctuating quantity $\hat{x}$ we consider a system which is in some definite stationary

state, say the n'th state. The average $\frac{1}{2}\overline{(\hat{x}_\omega \hat{x}_{\omega'} + \hat{x}_{\omega'} \hat{x}_\omega)}$ occurring in the equation $\frac{1}{2}\overline{(\hat{x}_\omega \hat{x}_{\omega'} + \hat{x}_{\omega'} \hat{x}_\omega)} = (x^2)_\omega \, \delta(\omega + \omega')$ is calculated as the corresponding diagonal matrix element

$$\frac{1}{2}(\hat{x}_\omega \hat{x}_{\omega'} + \hat{x}_{\omega'} \hat{x}_\omega) = \frac{1}{2}\sum_m [(x_\omega)_{nm}(x_{\omega'})_{mn} + (x_{\omega'})_{nm}(x_\omega)_{mn}] , \quad (3.27)$$

where the summation extends over the entire spectrum of energy levels. Because the operator $\hat{x}(t)$ is time dependent we must use the time-dependent wave functions to calculate its matrix elements. Hence

$$(x_\omega)_{nm} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x_{nm}\, e^{i\omega_{nm} t}\, e^{i\omega t}\, dt = x_{nm}\, \delta(\omega_{nm} + \omega) , \quad (3.28)$$

where $x_{nm}$ is the time-independent matrix element of the operator $\hat{x}$, expressed as a function of the coordinates of the particles of the system, and $\omega_{nm} = \frac{E_n - E_m}{\hbar}$. Using the fact that x is a real quantity and therefore the matrix elements satisfy the relation $x_{nm} = x^*_{mn}$ we find that

$$\frac{1}{2}(\hat{x}_\omega \hat{x}_{\omega'} + \hat{x}_{\omega'} \hat{x}_\omega) = \frac{1}{2}\sum_m |x_{nm}|^2 \, [\delta(\omega_{nm} + \omega)\, \delta(\omega + \omega') \quad (3.29)$$

$$+ \delta(\omega_{mn} + \omega)\, \delta(\omega + \omega')] .$$

On comparing this equation with equation (3.26) we obtain the result

$$(x^2)_\omega = \frac{1}{2}\sum_n |x_{nm}|^2 \, [\delta(\omega_{nm} + \omega) + \delta(\omega_{mn} + \omega)] . \quad (3.30)$$

Consider a periodic perturbation V with frequency $\omega$ acting on the system. Let the perturbation operator which represents the action of an external influence on the system have the form

$$\hat{V} = -f\hat{x} = -\tfrac{1}{2}(f_0\, e^{-i\omega t} + f_0^*\, e^{+i\omega t})\,\hat{x}\,. \qquad (3.31)$$

$\hat{x}$ is the quantum mechanical operator corresponding to the physical quantity x and f is a perturbing "force" given as a function of the time. Under the influence of this perturbation the system will make transitions away from the state n. From quantum mechanics, we have that the probability of a transition from a state n to a state m per unit time due to the absorption (or emission) of a quantum of energy $\hbar\omega_{nm}$ is

$$W_{nm} = \frac{\pi|f_0|^2}{2\hbar^2}\,|x_{nm}|^2 \left\{\delta(\omega+\omega_{mn}) + \delta(\omega+\omega_{nm})\right\}. \qquad (3.32)$$

The mean energy absorbed by the system per unit time is equal to

$$Q = \sum_m W_{nm}\,\hbar\,\omega_{mn}\,, \qquad (3.33)$$

$$= \frac{\pi}{2\hbar}|f_0|^2 \sum_m |x_{nm}|^2 \left\{\delta(\omega+\omega_{mn}) + \delta(\omega+\omega_{nm})\right\}\omega_{mn}\,,$$

$$= \frac{\pi}{2\hbar}\,\omega\,|f_0|^2 \sum_m |x_{nm}|^2 \left\{\delta(\omega+\omega_{nm}) - \delta(\omega+\omega_{mn})\right\}. \qquad (3.34)$$

Under the influence of the "force" f the system undergoes a change of state with an accompanying absorption of energy. The absorbed energy is dissipated into heat within the body. Under the action of a perturbation $\hat{V} = -\hat{x}f(t)$ the quantum mechanical average $\bar{x}$ does not vanish and can be represented in the form $\hat{\alpha} f$. $\hat{\alpha}$ is a linear integral operator whose effect on a function f(t) is given by an equation of the form

$$\bar{x}(t) = \int_0^\infty K(\tau)\, f(t-\tau)\, d\tau \,, \tag{3.35}$$

where K is a function of time which depends on the properties of the medium. The time-dependent perturbation can be expressed by means of a Fourier transformation, as a set of monochromatic components, whose time dependence is of the form $e^{-i\omega t}$. If we write $f(t-\tau) = f_0\, e^{-i\omega(t-\tau)}$ we obtain the relation

$$\bar{x} = \alpha(\omega)\, f \,, \tag{3.36}$$

where the function $\alpha(\omega)$ is given by

$$\alpha(\omega) = \int_0^\infty K(\tau)\, e^{i\omega\tau}\, d\tau \,. \tag{3.37}$$

If we know $\alpha(\omega)$ as a function of $\omega$ we will also know the behaviour of the system under the perturbation $\hat{V}$. $\alpha(\omega)$ is called the general susceptibility and plays an important role in the theory since it can be used to express the fluctuations in the physical quantity x.

In general, $\alpha(\omega)$ is a complex function and has the property

$$\alpha(-\omega) = \alpha^*(\omega) \,. \tag{3.38}$$

The real and imaginary parts of $\alpha(\omega)$ are denoted by $\alpha'$ and $\alpha''$ and because of equation (3.38) they satisfy the following equations:

$$\alpha'(-\omega) = \alpha'(\omega) \quad , \quad \alpha''(-\omega) = -\alpha''(\omega) . \tag{3.39}$$

If the function $\hat{H}(p,q,t)$ is the Hamiltonian of the system, the thermodynamic energy of the system can be written as $E = \overline{\hat{H}}(p,q,t)$. Using the result $\frac{dE}{dt} = \overline{\frac{\partial \hat{H}}{\partial t}}$ and the fact that the time dependence in the Hamiltonian occurs in the perturbation term we have

$$\frac{dE}{dt} = \overline{\frac{\partial \hat{V}}{\partial t}} = \frac{\partial}{\partial t}\left(-\overline{\hat{x}}\, f(t)\right) = -\bar{x}\,\frac{df}{dt} \;. \tag{3.40}$$

To find the average energy dissipation per second Q, we write f in the form

$$f = \frac{1}{2}\left(f_0\, e^{-i\omega t} + f_0^*\, e^{+i\omega t}\right) \tag{3.41}$$

and use the equation $\bar{x} = \alpha(\omega) f$ to obtain the result

$$\bar{x} = \frac{1}{2}\left[\alpha(\omega)\, f_0\, e^{-i\omega t} + \alpha(-\omega)\, f_0^*\, e^{i\omega t}\right] . \tag{3.42}$$

With the aid of eq.(3.41) and (3.42), we can write

$$\frac{dE}{dt} = -\overline{x\dot{f}} = -\frac{i\omega}{4}\Big(-\alpha(\omega)\, f_0^2\, e^{-2i\omega t} + \alpha(\omega)|f_0|^2 -$$

$$- \alpha(-\omega)|f_0|^2 + \alpha(-\omega)\, f_0^{*2}\, e^{+2i\omega t}\Big).$$

Averaging over one period of the external disturbance the terms containing $e^{\pm 2i\omega t}$ vanish and we are left with the result

$$Q = \frac{d\overline{E}^t}{dt} = \frac{i\omega}{4}(\alpha^* - \alpha)|f_0|^2 = \frac{\omega}{2}\alpha''|f_0|^2 \quad . \tag{3.43}$$

That is, the imaginary part of the general susceptibility function $\alpha(\omega)$ determines the energy lost to the system by absorption. Since there is always some energy absorption in any real process, we have that $Q > 0$, and we obtain the result that $\alpha''$ is always greater than zero and positive for positive frequencies. By comparing the two expressions for Q i.e. Eqn(3.34) and Eqn.(3.43) we obtain the result

$$\alpha''(\omega) = \frac{\pi}{\hbar}\sum_m |x_{nm}|^2 \left\{\delta(\omega + \omega_{nm}) - \delta(\omega + \omega_{mn})\right\} . \tag{3.44}$$

The two quantities $(x^2)_\omega$ and $\alpha''(\omega)$ are related to each other. To find this relation we use the result that the mean value of any physical quantity related to a given system can be calculated with the aid of the Gibb's distribution function from the formula

$$\bar{g} = \sum_n W_n^G g_{nn} = e^{F/kT}\sum_n f_{nn}\, e^{-E_n/kT} , \tag{3.45}$$

where $$W_n^c = \frac{e^{-E_n/kT}}{\Sigma\, e^{-E_n/kT}} = e^{F/kT}\, e^{-E_n/kT} ,$$

$E_n$ are the energy levels of the system,

$F$ is the free energy of the system.

In this way we find the average of $(x^2)_\omega$ to be

$$(x^2)_\omega = \frac{1}{2} \sum_n \sum_m e^{(F-E_n)/kT} \, |x_{nm}|^2 \left\{ \delta(\omega + \omega_{nm}) + \delta(\omega + \omega_{mn}) \right\} ; \quad (3.46)$$

interchanging n and m in the second term we get

$$= \frac{1}{2} \sum_n \sum_m e^{(F-E_n)/kT} \, (1 + e^{\hbar\omega_{nm}/kT}) \, |x_{nm}|^2 \, \delta(\omega + \omega_{nm}) \quad (3.47)$$

and using the property of the delta function we obtain the result

$$= \frac{1}{2} (1 + e^{-\hbar\omega/kT}) \sum_n \sum_m e^{(F-E_n)/kT} \, |x_{nm}|^2 \, \delta(\omega + \omega_{nm}) . \quad (3.48)$$

In a similar way we find the average of $\alpha''(\omega)$ to be

$$\alpha''(\omega) = \frac{\pi}{\hbar} (1 - e^{-\hbar\omega/kT}) \sum_n \sum_m e^{(F-E_n)/kT} \, |x_{nm}|^2 \, \delta(\omega + \omega_{nm}) , \quad (3.49)$$

By comparing these two expressions we obtain the important result

$$(x^2)_\omega = \frac{\hbar\alpha''}{2\pi} \coth \frac{\hbar\omega}{2kT} = \frac{\hbar\alpha''}{\pi} \left[ \frac{1}{2} + \frac{1}{e^{\hbar\omega/kT} - 1} \right] \quad (3.50)$$

and finally

$$\overline{x^2} = \frac{\hbar}{\pi}\int_0^\infty \alpha''(\omega) \coth\frac{\hbar\omega}{2kT}\, d\omega \, . \qquad (3.51)$$

If we regard the spontaneous fluctuations in the physical quantity x as due to the action of fictitious "random forces" f, we can obtain an expression for the spectral density of the mean square of the random forces. In order to accomplish this we treat x as a classical quantity and the eqn.(3.36), when written in terms of Fourier components, becomes:

$$x_\omega = \alpha(\omega)\, f_\omega \, . \qquad (3.52)$$

Eqn.(3.52) allows us to write the mean square fluctuations of $x_\omega$ in the form

$$\overline{x_\omega\, x_{\omega'}} = \alpha(\omega)\, \alpha(\omega')\, \overline{f_\omega f_{\omega'}} = (x^2)_\omega\, \delta(\omega+\omega') = |\alpha|^2 (f^2)_\omega\, \delta(\omega+\omega') \, . \qquad (3.53)$$

Finally using equation (3.50) we have the result

$$(f^2)_\omega = \frac{\hbar\alpha''}{2\pi|\alpha|^2} \coth\frac{\hbar\omega}{2kT} \, . \qquad (3.54)$$

If we are interested in several simultaneously fluctuating quantities $x_1$, $x_2$, $x_3$ ... the previous results can be immediately generalised.

Let $x_a$ and $x_b$ be any two of the set of $x_1$, $x_2$, ... $x_n$. The quantum-mechanical average of the symmetrised operator product

is defined by the equation

$$\frac{1}{2}\left(\hat{x}_{a\omega}\,\hat{x}_{b\omega'} + \hat{x}_{b\omega'}\,\hat{x}_{a\omega}\right) = (x_a\,x_b)_\omega\,\delta(\omega+\omega'). \tag{3.55}$$

By proceeding as in the derivation of eqn.(3.30) we get that

$$(x_a\,x_b) = \frac{1}{2}\sum_m\Big[(x_a)_{nm}\,(x_b)_{mn}\,\delta(\omega+\omega_{nm}) + (x_b)_{nm}\,(x_a)_{mn}\,\delta(\omega+\omega_{mn})\Big]. \tag{3.56}$$

The external perturbation operator is assumed to have the form:

$$\hat{V} = -\sum_a f_a\,\hat{x}_a = -\frac{1}{2}\sum_a\left\{f_{oa}\,e^{-i\omega t} + f^*_{oa}\,e^{i\omega t}\right\}\hat{x}_a\,. \tag{3.57}$$

The energy absorbed by the body is given by

$$Q = \frac{\pi}{2\hbar}\,\omega\sum_a\sum_b\sum_m f_{oa}\,f^*_{ob}\big[\,(x_a)_{mn}\,(x_b)_{nm}\,\delta(\omega+\omega_{nm}) -$$

$$- (x_a)_{nm}\,(x_b)_{mn}\,\delta(\omega+\omega_{mn})\big]\,. \tag{3.58}$$

Equation (3.42) is generalised to

$$\bar{x}_a = \frac{1}{2}\sum_b\left(\alpha_{ab}\,f_{ob}\,e^{-i\omega t} + \alpha^*_{ab}\,f^*_{ob}\,e^{+i\omega t}\right) \tag{3.59}$$

or

$$\bar{x}_a = \sum_b \alpha_{ab}\,f_b\,. \tag{3.60}$$

The rate of change of energy is given by

$$\frac{dE}{dt} = - \sum_{a} \dot{f}_a \, \overline{x}_a \tag{3.61}$$

and the energy dissipated is given by

$$Q = \frac{i\omega}{4} \sum_{a} \sum_{b} (\alpha^{*}_{ab} - \alpha_{ba}) \, f_{oa} \, f^{*}_{ob} \, , \tag{3.62}$$

Comparing equation (3.62) and (3.58) we find

$$\alpha^{*}_{ab} - \alpha_{ba} = - \frac{2\pi i}{\hbar} \sum_{m} [ \, (x_a)_{mn} \, (x_b)_{nm} \, \delta(\omega + \omega_{mn}) - - (x_a)_{nm} \, (x_b)_{mn} \, \delta(\omega + \omega_{mn}) \, ] \, , \tag{3.63}$$

The generalisation of equation (3.50) is

$$(x_a \, x_b)_\omega = \frac{i\hbar}{4\pi} (\alpha^{*}_{ba} - \alpha_{ab}) \coth \frac{\hbar\omega}{2kT} \; . \tag{3.64}$$

If we write

$$x_{a\omega} = \sum_{b} \alpha_{ab} \, f_{b\omega} \quad , \qquad f_{a\omega} = \sum_{b} \alpha^{-1}_{ab} \, x_{b\omega} \, , \tag{3.65}$$

and then substitute eqn.(3.64) into the following equation

$$(f_a \, f_b)_\omega = \sum_{c} \sum_{d} (\alpha^{-1}_{ac} \, \alpha^{-1*}_{bd}) \, (x_c \, x_d)_\omega \; . \tag{3.66}$$

we obtain the following generalized expression of eqn.(3.54):

$$(f_a f_b)_\omega = \frac{i\hbar}{4\pi} (\alpha^{-1}_{ab} - \alpha^{-1*}_{ba}) \coth \frac{\hbar\omega}{2kT} \quad . \tag{3.67}$$

These results can be applied to the problem of electromagnetic fluctuations. We outline the method used by Landau and Lifshits(6).

### 3.4 Application to Electromagnetic Fluctuations

The electric and magnetic induction fields are assumed to be of the form:

$$D_i = \hat{\varepsilon}_{ik} E_k + K_i \ , \tag{3.68}$$

$$B_i = \hat{\mu}_{ik} H_k + L_i \ ,$$

where the additional terms $K_i$ and $L_i$ represent the spontaneous local electric and magnetic moments per unit volume which arise as a result of the fluctuations in the position and motion of the charges in a body. Maxwell's equations as well as the above expressions for $\vec{D}$ and $\vec{B}$ are expressed in terms of their Fourier components. They take the following form:

$$D_{i\omega} = \varepsilon_{ik}(\omega) E_{k\omega} + K_{i\omega} \ , \tag{3.69}$$

$$B_{i\omega} = \mu_{ik}(\omega) H_{k\omega} + L_{i\omega} \ ,$$

$$(\nabla \times \vec{E}_\omega)_i = \frac{i\omega}{c} [\mu_{ik} H_{k\omega} + L_{i\omega}] \ , \tag{3.70}$$

$$(\nabla \times \vec{H}_\omega)_i = - \frac{i\omega}{c} [\varepsilon_{ik} E_{k\omega} + K_{i\omega}] \ . \tag{3.71}$$

In order to establish relations between the electromagnetic quantities appearing here and the quantities $x_a$ , $f_a$ it is assumed that K and L are not spontaneously arising moments but the result of an external action which places certain extraneous electric charges and currents in the body. Using the equation of conservation of energy, as it follows from Maxwell's equations, the resulting change in the energy of the body is calculated, i.e.

$$\int \frac{1}{4\pi}\left[\vec{E}\cdot\frac{\partial \vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial \vec{B}}{\partial t}\right] dV = -\frac{c}{4\pi}\oint \vec{E}\times\vec{H}\cdot d\vec{S} \, , \tag{3.72}$$

Substituting eqn.(3.68) into this relation gives

$$\int \frac{1}{4\pi}\left[ E_i \frac{\partial}{\partial t}(\hat{\varepsilon}_{ik} E_k) + H_i \frac{\partial}{\partial t}(\hat{\mu}_{ik} H_k)\right] dV$$

$$= -\frac{c}{4\pi}\oint (\vec{E}\times\vec{H})\cdot d\vec{S} - \frac{1}{4\pi}\int\left[\vec{E}\cdot\frac{\partial \vec{K}}{\partial t} + \vec{H}\cdot\frac{\partial \vec{L}}{\partial t}\right] dV \, .$$

The change in energy due to the "external action" amounts to

$$-\frac{1}{4\pi}\int\left[\vec{E}\cdot\frac{\partial \vec{K}}{\partial t} + \vec{H}\cdot\frac{\partial \vec{L}}{\partial t}\right] dV \, . \tag{3.73}$$

In order to reduce the continuous series of fluctuating quantities appearing here (the values of the fields at every point in the body) to a discrete series of quantities the volume of the body is divided into small but finite volume elements $\Delta V$ and the mean values of the fields in each volume element are regarded as a

discrete set of fluctuating quantities. This is equivalent to replacing the integral occurring in equation (3.73) by the sum:

$$- \frac{1}{4\pi} \sum_i \left[ \vec{E} \cdot \frac{\partial \vec{K}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{L}}{\partial t} \right]_i \Delta V_i \ , \tag{3.74}$$

where the summation runs over all volume elements $\Delta V_i$.

The following correspondence is then made:

$$\begin{aligned} x_a &\rightarrow \vec{E} \frac{\Delta V}{4\pi} \quad , \quad \vec{H} \frac{\Delta V}{4\pi} \\ f_a &\rightarrow \vec{K} \qquad , \quad \vec{L} \ . \end{aligned} \tag{3.75}$$

The expressions (3.65)

$$f_{a\omega} = \sum_b \alpha^{-1}_{ab}(\omega) \ x_{b\omega}$$

which give the relation between $x_\omega$ and $f_\omega$ correspond to Maxwell's equations (3.70), (3.71). These lead to the following relations:

$$K_{i\omega} = - \varepsilon_{ik} E_{k\omega} + \frac{ic}{\omega} (\nabla \times H_\omega)_i \ , \tag{3.76}$$

$$L_{i\omega} = - \mu_{ik} H_{k\omega} - \frac{ic}{\omega} (\nabla \times E_\omega)_i \ .$$

To find the coefficients $\alpha^{-1}_{ab}$ we compare equations (3.76) with equations (3.65) and use equation (3.75). The suffixes a and b enumerate both the components of $\vec{E}$, $\vec{H}$ and the volume elements $\Delta V$.

The curl operators in the equations (3.76) are to be considered as difference operators.

Equations (3.76) are valid for a given volume element. The coefficients $\alpha^{-1}_{ab}$ which relate the values of $\vec{K}$ to those of $\vec{H}$ as well as the coefficients relating the values of $\vec{L}$ to those of $\vec{E}$ satisfy the relation $\alpha^{-1}_{ab} = (\alpha^{-1}_{ba})^{*}$. Therefore from the equation

$$(f_a\, f_b)_\omega = \frac{i\hbar}{4\pi}\left(\alpha^{-1}_{ab} - \alpha^{-1*}_{ba}\right) \coth \frac{\hbar\omega}{2kT} \;,$$

we have

$$\left(K_{i(1)}\, L_{k(2)}\right)_\omega = 0 \;, \tag{3.77}$$

where the suffixes (1) and (2) signify that the quantities are to be evaluated in the volume elements which have their centres at $\vec{r}_1$ and $\vec{r}_2$, respectively. This result is valid both for $\vec{r}_1 = \vec{r}_2$ and for $\vec{r}_1 \neq \vec{r}_2$.

From the first equation (3.77) it is clear that the coefficients $\alpha^{-1}_{ab}$ which relate $K_{i\omega}$ and $E_{k\omega}$ are $-\,\varepsilon_{ik}\,\frac{4\pi}{\Delta V}$ if $\vec{K}_\omega$ and $\vec{E}_\omega$ refer to the same volume element and zero otherwise. Therefore by equation (3.67)

$$(f_a\, f_b)_\omega = \frac{i\hbar}{4\pi}\left(\alpha^{-1}_{ab} - \alpha^{-1*}_{ba}\right) \coth \frac{\hbar\omega}{2kT} \;,$$

we have

$$\left(K_{i(1)}\, K_{k(2)}\right)_\omega = 0 \;, \text{if } \vec{r}_1 \neq \vec{r}_2 \;,$$

$$(K_i K_k)_\omega = i\hbar(\varepsilon^*_{ki} - \varepsilon_{ik}) \frac{1}{\Delta V} \coth\frac{\hbar\omega}{2kT} .$$

If we now pass to the limit $\Delta V \to 0$, we can compress these relations into one equation.

$$\left(K_{i(1)} K_{k(2)}\right)_\omega = i\hbar(\varepsilon^*_{ki} - \varepsilon_{ik})\, \delta(\vec{r}_1 - \vec{r}_2) \coth\frac{\hbar\omega}{2kT} , \qquad (3.78)$$

where $\vec{r}_1$ and $\vec{r}_2$ refer to any two points in the body.

If the body is not in an external magnetic field we have $\varepsilon_{ik} = \varepsilon_{ki}$ and equation (3.78) can be written as follows:

$$\left(K_{i(1)} K_{k(2)}\right)_\omega = 2\hbar\, \varepsilon''_{ik}\, \delta(\vec{r}_2 - \vec{r}_1) \coth\frac{\hbar\omega}{2kT} . \qquad (3.79)$$

In a similar fashion we can derive the relation:

$$\left(L_{i(1)} L_{k(2)}\right)_\omega = 2\hbar\, \mu''_{ik}\, \delta(\vec{r}_2 - \vec{r}_1) \coth\frac{\hbar\omega}{2kT} . \qquad (3.80)$$

The presence of the delta function indicates that the fluctuations are correlated only in the limit when the two points coincide ($\vec{r}_2 \to \vec{r}_1$). This is to be taken in the macroscopic sense and means that the correlation extends only over regions which have been called "physically infinitesimal" volume elements.

For frequencies in the quasi-static region the tensor $\varepsilon_{ik}$ can be expressed in terms of a constant (frequency-independent) conductivity tensor $\sigma_{ik}$, i.e. by

$$\varepsilon_{ik} = \frac{4\pi i\,\sigma_{ik}}{\omega} \quad . \tag{3.81}$$

Define a current density $\vec{j} = \frac{1}{4\pi}\frac{\partial\vec{K}}{\partial t}$ . When expressed in terms of its Fourier components this relation becomes

$$\vec{j}_\omega = -\frac{i\omega\vec{K}_\omega}{4\pi} \quad . \tag{3.82}$$

Then we can write equation (3.71) in the form

$$(\nabla \times \vec{H}_\omega)_i = \frac{4\pi}{c}\left[\sigma_{ik}\, E_{k\omega} + j_{i\omega}\right] . \tag{3.83}$$

For quasi-static frequencies and for not too low temperatures we have $kT >> \hbar\omega$ and $\coth\frac{\hbar\omega}{kT} \approx \frac{2T}{\hbar\omega}$ . We can then write the relation

$$\left(K_{i(1)}\, K_{k(2)}\right)_\omega = 2\hbar\, \varepsilon''_{ik}\, \delta(\vec{r}_2 - \vec{r}_1) \coth\frac{\hbar\omega}{2kT} \tag{3.84}$$

in the following form:

$$\left(j_{i(1)}\, j_{k(2)}\right)_\omega = \frac{kT}{\pi}\,\sigma_{ik}\,\delta(\vec{r}_2 - \vec{r}_1) \ , \tag{3.85}$$

3.5 <u>Other Examples of Fluctuation Phenomena</u>

We now consider a number of examples of fluctuation phenomena whose theoretical analyses can be formulated in a way which is basically similar to that involved in Rytov's general theory of electromagnetic fluctuations. In each case the theoretical analysis will be simple and involve no complicated calculations; however the problems will involve a differential

equation, a fluctuating term and a correlation function. The discussion of these problems provides an introduction to the elementary treatment of fluctuation phenomena and should be of assistance in following the relatively more complex treatment of electromagnetic fluctuations as developed by Rytov.

<u>Brownian Motion</u>

As our first example of fluctuation theory we consider the perpetual irregular motion of small particles (dust particles, colloidal particles of radii $10^{-2}$ to $10^{-4}$ cm) suspended in a liquid.

The motions of these particles is due to the thermal agitation of the surrounding medium. The thermal agitation of the surrounding medium results in an unequal bombardment of the suspended particles on various sides by the molecule of the medium. Since there may be as many as $10^{21}$ collisions per second we cannot really speak of separate collisions.

We begin with Langevin's equation of motion for the suspended particles and consider only one-dimensional motion for simplicity. Langevin's equation is

$$m\dot{V} = -\frac{V}{B} + F(t) \quad \text{or} \quad \dot{V} = -\beta V + A(t) , \tag{3.86}$$

where: $m$ = mass of a suspended pole

$V = \dot{x}$ - velocity of the center of mass of the pole

$\beta = \frac{1}{mB}$ , $A(t) = \frac{F(t)}{m}$ .

This equation assumes that one can separate the influence of the surrounding medium on the motion of the particle into two parts:

(1) A fluctuating part $A(t)$ in which the discontinuity of events taking place is essential, and

(2) A systematic part $-\beta V$ in which the discontinuity of events is ignored. Because the suspended particles are much larger than the molecules, the surrounding medium is regarded as continuous and a frictional force proportional to the velcoity of the suspended particle is assumed to be valid. This is in accord with Stoke's law.

The function $A(t)$ is similar in nature to the functions K and M appearing in Rytov's theory, however it is only a function of the time t while K and M are functions of position and time.

A formal solution of the stochastic differential equation (3.86) can immediately be written down. It is

$$V(t) = V_o\, e^{-\beta t} + e^{-\beta t} \int_0^t A(\xi)\, e^{\beta\xi}\, d\xi \quad . \qquad (3.87)$$

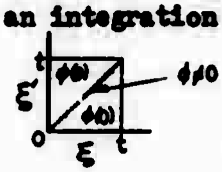
The quantity of interest is the mean square displacement $\overline{x^2}$. Squaring equation (3.87) we get

$$V^2(t) = V_o^2\, e^{-2\beta t} + e^{-2\beta t} \int_0^t\!\!\int_0^t A(\xi)A(\xi')\, e^{\beta(\xi+\xi')}\, d\xi d\xi' + 2V_o\, e^{-2\beta t} \int_0^t A(\xi) e^{\beta\xi}\, d\xi \quad .$$

We average over an ensemble of particles which we indicate by a horizontal bar and then using the fact that $\overline{A(\xi)} = 0$ (this is an assumption) we obtain the result:

$$\overline{V^2(t)} = \overline{V_0^2}\, e^{-2\beta t} + e^{-2\beta t} \int_0^t \int_0^t \overline{A(\xi)\, A(\xi')}\, e^{\beta(\xi+\xi')}\, d\xi\, d\xi' \,. \qquad (3.88)$$

We assume that $\overline{A(t)\, A(t')}$ depends only on the time interval and write $\phi(t'-t) = \overline{A(t)\, A(t')}$. That is, the correlation between the values of $A(t)$ at different times $t$ and $t'$ depends only on the interval of time elapsed. We assume further that $\phi(t'-t)$ has a sharp maximum at $t' = t$, i.e., it is large only for small time intervals. If we consider the accompanying diagram then $\phi(t'-t)$ is only large along a narrow region along the diagonal and is essentially zero everywhere else. To effect an integration we introduce the new variables




$$\xi' - \xi = s \quad , \quad \xi' + \xi = u \,, \qquad (3.89)$$

with $$d\xi\, d\xi' = \frac{ds\, du}{2} \,.$$

We extend the region of integration over $s$ from $-\infty$ to $+\infty$ since $\phi(s)$ is a rapidly decreasing function of $s$.

$$\overline{V^2(t)} = \frac{1}{2} \int_0^{2t} e^{\beta u} \int_{-\infty}^{+\infty} \phi(s)\, ds + \overline{V_0^2}\, e^{-2\beta t} \qquad (3.90)$$

$$= \int_{-\infty}^{+\infty} \phi\, ds\, \frac{(1 - e^{-2\beta(t)})}{2\beta} + \overline{V_0^2}\, e^{-2\beta t} \,.$$

The principle of equipartition of energy gives us the result that for $t \to \infty$, $\overline{V^2}(\infty) = \frac{kT}{m}$ .

$$\therefore \int_{-\infty}^{+\infty} \phi(s)\, ds = \frac{2kT\beta}{m} \; . \tag{3.91}$$

We can immediately find the correlation function for the velocity $V(t)$ from equation (2) by multiplying it by $V_o$ and averaging over an ensemble of particles. We obtain

$$\overline{V(t)V_o} = \overline{V_o^2}\, e^{-\beta|t|}$$

which we can write as

$$\overline{V(t)\, V(t')} = \overline{V^2}\, e^{-\beta|t-t'|} \; , \tag{3.92}$$

To find $\overline{x^2}$ we write $x = \int_0^t V(\xi)\, d\xi$ and hence

$$x^2 = \int_0^t\!\!\int_0^t V(\xi)\, V(\xi')\, d\xi\, d\xi' \; . \tag{3.93}$$

On averaging this over an ensemble of particles and using equation (3.92) we get the result

$$\overline{x^2} = \int_0^t\!\!\int_0^t \overline{V(\xi)\, V(\xi')}\, d\xi\, d\xi' = \overline{V^2} \int_0^t\!\!\int_0^t e^{-\beta|\xi-\xi'|}\, d\xi\, d\xi' \, .$$

Using the substitution (3.89)

$$\overline{x^2} = \frac{\overline{V^2}}{2} \int_0^{2t} du \int_{-\infty}^{+\infty} e^{-\beta|s|}\, ds \; , \tag{3.94}$$

$$\overline{x^2} = 2D\left[t - \frac{1}{\beta}\left(1 - e^{-\beta t}\right)\right] . \qquad \text{where } D = kTB$$

If a fluctuating force acts on a system which has a frequency-dependent structure it is important to know the frequency or spectral distribution of the correlation function. Let $A(t)$ be any fluctuating or statistical function, and let the average value $\overline{A(t)}$, taken over long time intervals be zero, that is, assume

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T A(t)\, dt = 0 \quad . \tag{3.95}$$

We also assume that $A(t)$ is different from zero only in a finite but a very large time-interval, i.e.

$$A(t) = 0 \qquad \text{for } t < 0 \quad \text{and} \quad t > T \; . \tag{3.96}$$

At the end of our calculations we can let $T$ go to infinity. We expand $A(t)$ as a Fourier integral by writing

$$A(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} C(\omega)\, e^{i\omega t}\, d\omega \quad , \tag{3.97}$$

$$\text{where: } C(-\omega) = C^*(\omega) \; . \tag{3.98}$$

By Plancherel's theorem[8] we have

$$\int_{-\infty}^{+\infty} A^2(t)\, dt = \int_{-\infty}^{+\infty} |C(\omega)|^2\, d\omega \tag{3.99}$$

or dividing both of this by T we have

$$\overline{A^2} = \int_{-\infty}^{+\infty} \lim_{T\to\infty} \frac{|C(\omega)|^2}{T} \, d\omega \, . \qquad (3.100)$$

Since $|C(\omega)|^2$ is symmetric in $\omega$, we can write:

$$\overline{A^2(t)} = \int_0^{\infty} A_\omega^2 \, d\omega \qquad (3.101)$$

with

$$A_\omega^2 = \lim_{T\to\infty} \left[ \frac{2}{T} |C(\omega)|^2 \right] . \qquad (3.102)$$

We regard equation (3.102) as defining the spectral distribution of $\overline{A^2}$. The statistical behaviour of A(t) is characterised by the correlation function

$$\phi(\tau) = \overline{A(t)\, A(t+\tau)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{+T} A(t)\, A(t+\tau)\, dt \, . \quad (3.103)$$

The correlation function is a measure of how that value of A at some time t affects the probabilities of different values of A at a time $t+\tau$. The bar indicates an averaging over different times t for a fixed value of $\tau$. To find the relation between $\phi(\tau)$ and the spectral distribution $A_\omega^2$ as given by equation (3.101) we consider

$$\int_{-T}^{+T} A(t)\, A(t+\tau)\, dt = \frac{1}{2\pi} \iiint C(\omega)\, C^*(\omega')\, e^{i(\omega-\omega')t}\, e^{-i\omega' t}\, d\tau\, d\omega\, d\omega' .$$

Using the fact that

$$\frac{1}{2\pi}\int_{-T}^{+T} e^{-i(\omega-\omega')t}\, dt = \delta(\omega-\omega') ,$$

we get on dividing by T and taking the limit as $T \to 0$

$$\phi(\tau) = \int_0^\infty A_\omega^2 \cos\omega\tau \, d\omega . \tag{3.104}$$

(since $A_\omega^2$ is an even function of $\omega$ , $\int A_\omega^2 \sin\omega\tau \, d\omega = 0$).

We see at once

$$\phi(0) = \overline{A^2} = \int_0^\infty A_\omega^2 \, d\omega . \tag{3.105}$$

We have also

$$A_\omega^2 = \frac{1}{\pi}\int_{-\infty}^{+\infty} \phi(\tau) \cos\omega\tau \, d\tau . \tag{3.106}$$

That is, the correlation function $\phi(\tau)$ and the spectral distribution $A_\omega^2$ are Fourier transforms of each other.

As an example of the fruitfulness of introducing the spectral distribution function $A_\omega^2$ , let us reconsider the Langevin equation for Brownian motion:

$$\dot{V} + \beta V = A(t) . \tag{3.107}$$

If we expand A(t) as a Fourier integral we find at once from equation (3.107)

$$V(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{C(\omega)}{\beta + i\omega} e^{i\omega t} d\omega .$$

From equations (3.101) and (3.102) we have

$$\overline{V^2(t)} = \int_0^{\infty} V_\omega^2 \, d\omega ,$$

$$V_\omega^2 = \frac{A_\omega^2}{\beta^2 + \omega^2} .$$

Using the result

$$\overline{V(t)\, V(t+\tau)} = \overline{V^2}\, e^{-\beta|\tau|} \tag{3.108}$$

(this is given by equation (3.92) in the discussion of Brownian motion) we have from equation (3.106)

$$V_\omega^2 = \frac{\overline{V^2}}{\pi} \int_{-\infty}^{+\infty} e^{-\beta|\tau|} \cos \omega\tau \, d\tau = \frac{2}{\pi} \overline{V^2} \frac{\beta}{\beta^2 + \omega^2} . \tag{3.109}$$

Since $m\overline{V^2} = kT$ we have that

$$A_\omega^2 = \frac{2}{\pi} \frac{kT}{m} \beta . \tag{3.110}$$

Since $A_\omega^2$ does not depend on the frequency we have that $\overline{A^2} = \int A_\omega^2 \, d\omega$ becomes infinitely large. Hence we can conclude that our initial equation is of limited validity. In particular, the assumption that the frictional force is given by $-\beta V$ where $\beta$ is constant must

be incorrect if the frequency of the exciting force can increase indefinitely. Therefore equation (3.86) is only valid up to a certain cut-off frequency which is determined by the model used. This difficulty can be removed by assuming $\beta(\omega)$.

Thermal Radiation

It is possible to find the spectral distribution of the energy density of an isotropic thermal radiation field by allowing the radiation field to act on a weakly damped harmonic oscillator.

We consider a weakly-damped simple harmonic oscillator acted on by a fluctuating face $A(t)$. The equation of motion of the oscillator of mass m and charge e is given by

$$\ddot{x} + \beta(\omega)\dot{x} + \omega_0^2 x = A(t) \; . \tag{3.111}$$

We assume the damping force $-\beta(\omega)\dot{x}$ to be frequency-dependent. The damping will be weak if $\omega_0^2 >> \frac{\beta^2}{4}$. We write $A(t)$ and $x(t)$ as Fourier-integrals:

$$A(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} C(\omega)\, e^{i\omega t}\, dt$$

$$x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \xi(\omega)\, e^{i\omega t}\, dt \; .$$

From the equation of motion (3.111) we have

$$\xi(\omega) = \frac{C(\omega)}{-\omega^2 + \omega_0^2 + i\,\omega\,\beta(\omega)} \; . \tag{3.112}$$

The velocity $V = \dot{x}$ is given by:

$$V(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{i\omega C(\omega)\, e^{i\omega t}}{\omega_0^2 - \omega^2 + i\omega\beta(\omega)}\, d\omega \; . \tag{3.113}$$

According to equations (3.101) and(3.102) we can write

$$\overline{V^2} = \int_0^{\infty} V_\omega^2 \, d\omega$$

where

$$V_\omega^2 = \frac{A_\omega^2 \, \omega^2}{(\omega^2 - \omega_0^2)^2 + \omega^2\beta^2(\omega)} \tag{3.114}$$

$$\text{i.e.} \quad \overline{V^2} = \int_0^{\infty} \frac{A_\omega^2 \, \omega^2 \, d\omega}{(\omega^2 - \omega_0^2)^2 + \omega^2\beta^2(\omega)} \; . \tag{3.115}$$

A simple treatment of this equation is only possible for the case of weak damping. We assume that $A_\omega^2$ and $\beta(\omega)$ are smoothly varying functions of $\omega$ and since the maximum contribution to the integral comes from the region near $\omega = \omega_0'$ , we write

$$\overline{V^2} = A_{\omega_0}^2 \int_0^{\infty} \frac{\omega^2 \, d\omega}{(\omega^2 - \omega_0^2)^2 + \omega^2\beta^2(\omega_0)} = A_{\omega_0}^2 \, \frac{\pi}{2\beta(\omega_0)} \; . \tag{3.116}$$

We interpret equation (3.111) in the following way. We start with an undamped harmonic oscillator whose equation of motion is $\ddot{x} + \omega_0^2 x = 0$ and bring it into contact with a macroscopic medium at a temperature T. The nature of the contact with the

medium determines $\beta(\omega)$ and $A(t)$. In this way we can assume that the relation given by equation (3.116) is valid for all frequencies and write

$$\overline{V^2} = A_\omega^2 \frac{\pi}{2\beta(\omega)} \; . \qquad (3.117)$$

Now $\overline{mV^2}$ = average total energy $\varepsilon(T)$ where:

$$\varepsilon(T) = kT \quad \text{classical physics} , \qquad (3.118)$$

$$\varepsilon(T) = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1} \quad \text{quantum physics} . \qquad (3.119)$$

We can write:

$$A_\omega^2 = \frac{2\beta(\omega)\,\varepsilon(T)}{\pi m} \; . \qquad (3.120)$$

For a particle of mass $m$ and charge $e$ in a fluctuating field $\vec{E}$, whose x-component is $E(t)$ we have:

$$A(t) = \frac{e\,E(t)}{m} \qquad (3.121)$$

and therefore we can write:

$$E_\omega^2 = \frac{m^2}{e^2} A_\omega^2 = \varepsilon(T)\,\frac{2}{\pi}\,\frac{m}{e}\,\beta(\omega) \, . \qquad (3.122)$$

For an oscillator oscillating with frequency $\omega$ we have (from classical electromagnetic theory) that the damping coefficient $\beta(\omega)$ is equal to $\frac{2}{3}\frac{e^2\omega^2}{mc^3}$ . This damping is due to the radiation of

energy by the oscillator.

Since the oscillator is in thermal equilibrium with a medium at temperature T, it must have an average energy given by equation (3.115) or (3.119). Hence to make up for the loss of energy there must be present an incident radiation field whose spectral distribution is given by:

$$E_\omega^2 = \varepsilon(T)\,\frac{2}{\pi}\,\frac{m}{e^2}\,\beta(\omega) = \varepsilon(T)\,\frac{4}{3\pi}\,\frac{\omega^2}{C^3}\,. \qquad (3.123)$$

The average energy density of thermal radiation is given by

$$\overline{u^T} = \frac{\overline{E^2 + H^2}}{8\pi}\,. \qquad (3.124)$$

For isotropic radiation we have

$$\overline{E_x^2} = \overline{E_y^2} = \overline{E_z^2} = \overline{H_x^2} = \overline{H_y^2} = \overline{H_z^2}\,,$$

that is,

$$\overline{u^T} = \frac{6}{8\pi}\,\overline{E_x^2}\,.$$

Therefore the spectral distribution of the energy density must be

$$u_\omega = \frac{6}{8\pi}\,\varepsilon(T)\,\frac{4}{3\pi}\,\frac{\omega^2}{C^3} = \frac{\omega^2}{\pi^2 C^3}\left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1}\right]. \qquad (3.125)$$

$u_\omega\, d\omega$ is the incident energy density of the thermal radiation field in the frequency interval between $\omega$ and $\omega + d\omega$. We can, if we wish, think of the walls of a cavity enclosing thermal radiation

as a collection of damped harmonic oscillators.

By considering the oscillator between the two plates of a capacitor (separation distance a) joined by a resistance R it is possible to show that there exists a damping given by $\beta = \frac{e^2}{a^2 m} R$. This damping is due to the Joule's heat loss. One can show $\frac{1}{2} C\overline{V^2} = \frac{kT}{2}$ where C is the capacity of the capacitor and V is the fluctuating voltage across the capacitor. Nyquist's formulae can also be derived in this way.

## IV METHOD OF SOLUTION

The procedure for solving specific problems of equilibrium radiation will be outlined in this section. It will also be shown how Rytov proceeds to evaluate the constant arising in the correlation function for the fluctuating field.

In the study of equilibrium radiation from a bounded medium, the first step is to find a solution for the primary fields within the medium, $E_0$ and $H_0$ which depend on the random field K and which satisfy the basic equations (2.89):

$$\text{Curl } \vec{E}_0 = - ik\mu \vec{H}_0$$

$$\text{Curl } \vec{H}_0 = ik\varepsilon \vec{E}_0 + ik(\varepsilon - 1)\vec{K} \qquad (4.1)$$

where it is assumed that there are no magnetic losses in the medium ($\vec{M} = 0$).

An expression is then written for the fields $E_r$ and $H_r$ which represent the reflection within the medium of the primary fields $E_0$ and $H_0$. At the boundaries $E_r$ and $H_r$ must satisfy the basic equations (2.89) with $K = 0$, i.e.

$$\text{Curl } \vec{E}_r = - ik\mu \vec{H}_r$$

$$\text{Curl } \vec{H}_r = ik\varepsilon \vec{E}_r \, . \qquad (4.2)$$

The fields E and H in free-space outside the bounded medium are then assumed to satisfy the basic equations in vacuum,

that is, with $\mu = \varepsilon = 1$

$$\text{Curl}\ \vec{E} = -\,ik\vec{H}$$

$$\text{Curl}\ \vec{H} = ik\vec{E}\ , \tag{4.3}$$

The general solution of these various sets of differential equations may be written in the form of a Fourier series expansion. In general the solution for the electric field may be written in the form of

$$\vec{A}(r) = \sum_{n=-\infty}^{+\infty} \vec{a}_n(\vec{p})\ e^{i\bar{p}_n\cdot\bar{r}}$$

$$= \sum_{n_1}\sum_{n_2}^{+\infty}\sum_{n_3} a_n(p)\ e^{i(p_{n_1}x_1 + p_{n_2}x_2 + p_{n_3}x_3)} \tag{4.4a}$$

where $\vec{p}_n$ is the propagation vector $= (2\pi/\lambda_n)\vec{p}$

$\vec{p}$ is a unit vector in the direction of propagation

$\lambda_n$ is the wavelength of the $n^{th}$ wave in the expansion

$x_1, x_2, x_3$ are the projections of r along three Cartesian axes

$p_{n_1}, p_{n_2}, p_{n_3}$ are the projections of $\vec{p}$ along the same axis.

In problems where the medium extends indefinitely along a given direction, it is useful to write the general solution as a Fourier integral expansion.

$$\vec{A}(\vec{r}) = \iiint_{-\infty}^{+\infty} a(\vec{p})\, e^{i(p_1x_1 + p_2x_2 + p_3x_3)}\, dp_1\, dp_2\, dp_3$$

or

$$\vec{A}(\vec{r}) = \int_{-\infty}^{+\infty} \vec{a}(\vec{p})\, e^{i\vec{p}\cdot\vec{r}}\, d\vec{p} \qquad (4.4b)$$

where $\qquad d\vec{p} = dp_1\, dp_2\, dp_3$ .

In particular this kind of expansion is used for expressing the random field $\vec{K}$. The expression for $\vec{a}(p)$ is obtained from the Fourier transform of $A(\vec{r})$:

$$\vec{a}(\vec{p}) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{+\infty} \vec{A}(\vec{r})\, e^{-i(p_1x_1 + p_2x_2 + p_3x_3)}\, dx_1\, dx_2\, dx_3$$

$$\vec{a}(\vec{p}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} \vec{A}(r)\, e^{-i\vec{p}\cdot\vec{r}}\, d\vec{r} \, . \qquad (4.5)$$

When using the integral expansion for $\vec{K}$, the same restriction as for the expansion (Eqn.3.4) applies. This type of expansion must be regarded as a formal representation of a Fourier series.

For the three sets of differential equations (4.1)(4.2)(4.3) the solution for the electric vector can then be expressed in the following way. The primary field

$$E_e(r) = \sum_{n=-\infty}^{+\infty} a_n(p)\, e^{i\vec{p}_n\cdot\vec{r}} \tag{4.6}$$

$$K(r) = \sum_{n=-\infty}^{+\infty} g_n(p)\, e^{i\vec{p}_n\cdot\vec{r}} = \int_{-\infty}^{+\infty} g_n(p)\, e^{i\bar{p}\cdot\bar{r}}\, d\bar{p} \tag{4.7}$$

For the reflected field inside the medium

$$E_r(r) = \sum_{n=-\infty}^{+\infty} b_n(p)\, e^{i\vec{s}_n\cdot\vec{r}} \tag{4.8}$$

For the transmitted field outside the medium

$$E(r) = \sum_{n=-\infty}^{+\infty} c_n(p)\, e^{i\vec{t}_n\cdot\vec{r}} \tag{4.9}$$

$\vec{p}_n$ , $\vec{s}_n$ and $\vec{t}_n$ describe the direction and magnitude of the propagation vectors of the various waves. They are in general related by some constants depending on the geometry and the electromagnetic properties $(\varepsilon,\mu)$ of the medium. The coefficients $a_n$ , $b_n$ and $c_n$ may contain more than one term for the case of waves propagating in opposite directions.

The expression for the magnetic field vector equation is then obtained by means of the first equation (Curl $\vec{E}$) for each set of fundamental equations (4.1),(4.2),(4.3). Using Eqns. (4.6), (4.8), and (4.9) this leads to

$$H_0(r) = -\frac{1}{k\mu}\sum_n (\vec{p}_n \times \vec{a}_n)\, e^{i\vec{p}_n\cdot\vec{r}} \tag{4.10}$$

$$\vec{H}_r(\bar{r}) = -\frac{1}{k\mu}\sum_n (\vec{s}_n \times \vec{b}_n)\, e^{+i\vec{s}_n\cdot\bar{r}} \tag{4.11}$$

$$\vec{H}(\bar{r}) = -\frac{1}{k}\sum_n (\vec{t}_n \times \vec{c}_n)\, e^{i\bar{t}_n\cdot\bar{r}} \tag{4.12}$$

By means of the second equation in (4.1) and the equations (4.6),(4.7) and (4.10) one obtains an expression for the coefficient $\vec{a}_n$ in terms of $\vec{e}_n$.

The coefficients $\vec{a}_n$, $\vec{b}_n$ and $\vec{c}_n$ are then evaluated in applying the boundary conditions (11),

$$\begin{aligned} \vec{n} \times (\vec{E}_0 + \vec{E}_1) &= \vec{n} \times \vec{E} \\ n \times (\vec{H}_0 \times \vec{H}_1) &= \vec{n} \times \vec{H} \end{aligned} \tag{4.13}$$

and the conditions that at the boundary the phase of the waves is equal

$$\vec{p}_n\cdot\vec{r} = \vec{s}_n\cdot\vec{r} = \vec{t}_n\cdot\vec{r}. \tag{4.14}$$

and that div $E_r = 0$ and div $E = 0$, i.e.

$$\begin{aligned} \vec{a}_n \cdot \vec{b}_n &= 0 \\ \vec{t}_n \cdot \vec{c}_n &= 0 \end{aligned} \tag{4.15}$$

The solution of the six equations obtained from the boundary conditions allows an explicit representation of $\vec{c}_n$ in terms of $\vec{\mathcal{E}}_n$ and the other constants of the system.

The intensity of radiation outside the bounded medium is obtained by calculating the space and time-average of the Poynting vector. This reduces the problem to the evaluation of the space-average of the expression (3.7 ). Thus

$$\bar{S} = \pi I_\nu = \frac{c}{4\pi}\left[\overline{E \times H^*} + \overline{E^* \times H}\right] \tag{4.16}$$

where $I_\nu$ is the intensity of radiation and the bar over the product indicates the space average of the product of the $\vec{E}$ and $\vec{H}$ vectors. The final expression to be calculated will be of the form, using (4.16),(4.9),(4.12),

$$\overline{E \times H^*} = \frac{1}{k^2\mu} \sum_{n'} \sum_{n''} \overline{\vec{c}_{n'}(p) \times \vec{t}^*_{n''} \times \vec{c}^*_{n''}(p)}$$

and since $\vec{c}_n(p)$ can be expressed in terms $\vec{\mathcal{E}}_n(p)$ the final operation consists in evaluating the space average of the product $\overline{\vec{\mathcal{E}}_{n'}(p) \times \vec{\mathcal{E}}^*_{n''}(p')}$ , that is, the space correlation function $G_{n'n''}(p) = \overline{\vec{\mathcal{E}}_{n'}(p') \times \vec{\mathcal{E}}^*_{n''}(p')}$ or using Rytov's notation[1]

$$G_{\alpha\beta}(p',p'') = G_{\alpha\beta}(p) = \overline{\vec{\mathcal{E}}_\alpha(p')\, \mathcal{E}^*_\beta(\vec{p}'')} \tag{4.17}$$

Using the integral representation for $\vec{K}$ (Eqn. 47)

$$\vec{K}(r) = \int_{-\infty}^{+\infty} \vec{g}(p)\, e^{i\bar{p}\cdot\bar{r}}\, d\bar{p}$$

$$\vec{g}(p) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} u(\vec{r})\, e^{-i\bar{p}\cdot\bar{r}}\, d\vec{r}$$

and the correlation function, Eqn.(3.12)

$$F_{\alpha\beta}(r) = C\delta_{\alpha\beta}\, \delta(\vec{r}) = \overline{\vec{K}'(r')\, K^*(r'')}$$

the following expression is obtained for the correlation $G_{n'n''}(p)$

$$G_{\alpha'\beta''}(\vec{p}) = \frac{C\delta_{\alpha\beta}}{(2\pi)^3}\, \delta(\vec{p}) \tag{4.18}$$

In the case of a finite correlation radius as defined by Eqn.(3.13) and (3.10), the function $\vec{G}_{\alpha\beta}(p)$ becomes

$$G_{\alpha'\beta'}(p) = \frac{C\delta_{\alpha\beta}}{(2\pi)^3}\, e^{-a^2p^2/4}\, \delta(\vec{p}) \tag{4.19}$$

which for $a \to 0$ gives Eqn.(4.18).

To solve the problem numerically it is still necessary to evaluate the constant C in the correlation function (4.18). The evaluation of C is obtained by studying the flow of equilibrium

radiation from a semi-infinite space bounded by an infinite plane. The results are then compared with Kirchoff's law (see Rytov).[1]

$$I_\omega = I_{o\omega}(1 - R)$$

for which there is no restriction concerning the absorptivity in the medium.

$I_\omega$ is the intensity of radiation in the medium

$I_{o\omega}$ is the intensity of radiation in free-space, and

$R$ is the reflectivity for the $I_{o\omega}$ incident on the interface.

From this comparison Rytov obtains the following value for the constant C.

$$C = \frac{16\pi^3}{k^2 c} I_{o\omega} I_m \left(\frac{1}{\varepsilon - 1}\right) \tag{4.20}$$

It will be shown in the next section that the use of the constant C (Eqn. 4.20) in the evaluation of the equilibrium radiation from a cylinder leads to the expected results for the limiting case where the geometrical optics theory is valid.

## V RADIATION FROM A CYLINDRICAL PLASMA

The equilibrium radiation from a cylindrical plasma of radius "a" and of infinite length will be derived using Rytov's theory of thermal radiation[1] as outlined above. Such a solution is important not only for its possible applications but also for its theoretical implications. In fact, the present derivation provides an exact treatment to the problem of radiation from cylindrical bodies including such effects as internal reflections, polarisation and diffraction. It is also valid for any range of frequencies of the radiation spectrum and for any size of cylinders. The practical aspect of this problem is due to the fact that it describes a valuable model for the study of equilibrium radiation from the wake of a re-entry vehicle. In deriving the solution for the emission of radiation by a cylindrical body it will also be shown that the solution may be expressed in terms of the power absorptivity of the body.

In this problem the primary fields $E_0$, $H_0$, the reflected fields $E_r$, $H_r$ and the transmitted fields $E$, $H$ have to satisfy respectively the basic differential equations (4.1), (4.2) and (4.3) written in cylindrical coordinates. Since the solution will involve the description of waves propagating from a cylinder, the independent fundamental solutions may be written in terms of the three basic vector solutions ($\vec{M}$, $\vec{N}$ and $\vec{L}$) of the wave equation (Stratton)[9]. Rytov writes these particular independent solutions,

using the cylindrical coordinates $r, \phi, z$ in the following way:

$$\vec{M}_{nh}(r,\phi,z) = e^{i(n\phi+hz)}\left(\frac{i\,n\,Z_n}{r}\,\vec{i}_1 - Z_n'\,\vec{i}_2\right) \tag{5.1a}$$

$$\vec{N}_{nh}(r,\phi,z) = e^{i(n\phi+hz)}\left(\frac{i\,h\,Z_n'}{q}\,\vec{i}_1 - \frac{h\,n\,Z_n}{qr}\,\vec{i}_2 + \frac{\lambda^2 Z_n}{q}\,\vec{i}_3\right) \tag{5.1b}$$

$$\vec{L}_{nh}(r,\phi,z) = e^{i(n\phi+hz)}\left(Z_n'\,\vec{i}_1 + \frac{i\,n\,Z_n}{r}\,\vec{i}_2 + i\,h\,Z_n\,\vec{i}_3\right) \tag{5.1c}$$

where $\vec{i}_1$, $\vec{i}_2$ and $\vec{i}_3$ are the unit vectors related to the coordinates $r, \phi, z$ respectively. $Z_n = Z_n(\lambda r)$ is any cylindrical function of order n. $Z_n'$ is the derivative of $Z_n$ with respect to r.

$$\lambda = \sqrt{q^2 - h^2} \tag{5.2}$$

where $q = k\sqrt{\varepsilon\mu}$ is the propagation constant inside the cylinder. In free-space $q = k$ and $\lambda = \lambda_0 = \sqrt{k^2 - h^2}$ .

The functions given in Eq.(5.1) have the following properties:

$$\begin{aligned} \text{Curl } \vec{M}_n &= q\vec{N}_n \\ \text{Curl } \vec{N}_n &= q\vec{M}_n \\ \text{Curl } \vec{L}_n &= 0 \end{aligned} \tag{5.3a}$$

$$\begin{aligned} \text{div } \vec{M}_n &= 0 \\ \text{div } \vec{N}_n &= 0 \end{aligned} \tag{5.3b}$$

In the case of equations (4.2) and (4.3), for which $\operatorname{div} E_r = \operatorname{div} E = 0$, the particular solution will involve a linear combination of the $\vec{M}_n$ and $\vec{N}_n$ functions only. For the equations (4.1) ($\operatorname{div} E_o \neq 0$), the three functions $\vec{M}_n$, $\vec{N}_n$ and $\vec{L}_n$ will enter in the solution. The general solution for each set of equations (4.1), (4.2), (4.3) will then be represented by a linear expansion (over n and h) of each of the functions $\vec{M}_n$, $\vec{N}_n$ and $\vec{L}_n$. It is then possible to determine the coefficients of these expansions because of the orthogonal properties of $\vec{M}_n$, $\vec{N}_n$ and $\vec{L}_n$.

The particular solutions for the electric fields $\vec{E}_o$, $\vec{E}_r$ and $\vec{E}$ can then be written as:

$$
\begin{aligned}
\vec{E}_{on} &= A_n\vec{M}_n + B_n\vec{N}_n + C_n\vec{L}_n + \tilde{A}_n\vec{\tilde{M}}_n + \tilde{B}_n\vec{\tilde{N}}_n + \tilde{C}_n\vec{\tilde{L}}_n \\
\vec{E}_{rn} &= A_{rn}\vec{M}_n + B_{rn}\vec{N}_n \qquad\qquad (5.4) \\
\vec{E}_n &= P_n\vec{M}^{\circ}_n + Q_n\vec{N}^{\circ}_n
\end{aligned}
$$

where: $A_n$, $B_n$, $C_n$, $A_{rn}$, $B_{rn}$, $P_n$, $Q_n$ are coefficients depending on n, h and r.

$\vec{M}_n$, $\vec{N}_n$ and $\vec{L}_n$ contain cylindrical functions of the type $Z_n = J_n(\lambda r)$ (Bessel function).

$\vec{\tilde{M}}_n$, $\vec{\tilde{N}}_n$ and $\vec{\tilde{L}}_n$ contain $Z_n = N_n(\lambda r)$ (Neuman function), and

$\vec{M}^{\circ}_n$ and $\vec{N}^{\circ}_n$ contain $Z_n = H_n^{(2)}(\lambda_o r)$ (Hankel functions of the second kind).

The general solution is represented by a linear combination of the particular solutions. The orthogonal properties of $\vec{M}_n$, $\vec{N}_n$ and $\vec{L}_n$ then permit an evaluation of the coefficients in these expansions.

The general solutions for the various fields inside and outside the cylinder are expressed as follows:

The primary field:

$$\vec{E}_o = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left( A\vec{M}_n + B\vec{N}_n + C\vec{L}_n + A\widetilde{\vec{M}}_n + B\widetilde{\vec{N}}_n + C\widetilde{\vec{L}}_n \right) dh \quad (5.5a)$$

$$\vec{H}_o = i\sqrt{\frac{\varepsilon}{\mu}} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left( A\vec{N}_n + B\vec{M}_n + A\widetilde{\vec{N}} + B\widetilde{\vec{M}} \right) dh \quad (5.5b)$$

The reflected field:

$$\vec{E}_r = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left( A_r\vec{M}_n + B_r\vec{N}_n \right) dh \quad (5.6a)$$

$$\vec{H}_r = i\sqrt{\frac{\varepsilon}{\mu}} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left( A_r\vec{N}_n + B_r\vec{M}_n \right) dh \quad (5.6b)$$

The field outside the medium

$$\vec{E} = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} (P_n \vec{M}_n^e + Q_n \vec{N}_n^e)\, dh \tag{5.7a}$$

$$\vec{H} = i \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} (P\vec{N}_n^e + Q\vec{M}_n^e)\, dh \tag{5.7b}$$

The summation over n has been transformed into an integral expansion since the cylinder extends from $-\infty$ to $+\infty$ along the z-axis. The expression for the H-field has been deduced from the equation for the E-field by means of equations (4.1), (4.2) and (4.3). The determination of the coefficients $A_n$, $B_n$ ... $P_n$, $Q_n$ as function of r is carried out in the Appendix.

As indicated in a previous section the random field $\vec{K}$ is expressed by an infinite expansion in terms of the function $\vec{G}_{nh}(r)$

$$\frac{\varepsilon - 1}{\varepsilon} \vec{K} = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(n\phi + hz)}\, \vec{G}_{nh}(r)\, dh \tag{5.8}$$

and its Fourrier transform is

$$\vec{G}_{nh}(r) = \frac{(\varepsilon - 1)}{(2\pi)^2 \varepsilon} \int_{-\infty}^{+\infty} dz \int_{-\pi}^{+\pi} \vec{K} e^{-i(n\phi + hz)}\, d\phi \tag{5.9}$$

On the surface of the cylinder, the boundary conditions (4.13) take the form of

$$\left.\begin{array}{lll} E_{e\phi} + E_{r\phi} = E_\phi & ; & E_{ez} + E_{rz} = E_z \\ H_{e\phi} + H_{r\phi} = H_\phi & ; & H_{ez} + H_{rz} = H_z \end{array}\right| (r=a) \qquad (5.10)$$

From these conditions it is possible to express P and Q in terms of $\tilde{A}$ and $\tilde{B}$ which, in turn can be expressed by the components of $\vec{G}$ given by Eq.(5.9). The correlation function for the field $\vec{K}$ in cylindrical coordinates is written as, (using Eqns.(3.8) and (3.12):

$$F_{\alpha\beta} = \overline{K_\alpha(r,\phi,z)\, K_\beta^*(r_1,\phi_1,z_1)}$$

$$= c\,\delta_{\alpha\beta}\,\delta(r-r_1)\,\frac{\delta(\phi-\phi_1)}{r}\,\delta(z-z_1) \qquad (5.11)$$

where: $\alpha, \beta = r,\phi,z$

From (5.9) and (5.11) one obtains the correlation function for $\vec{G}_{nh}$:

$$\overline{G_{nh\alpha}(r)\, G_{mh_1\beta}(r_1)} = \frac{|\varepsilon-1|^2}{4\pi^2|\varepsilon|^2}\, C\,\delta_{nm}\,\delta_{\alpha\beta}\,\frac{\delta(r-r_1)}{r}\,\delta(h-h_1) \qquad (5.12)$$

where the constant C is given by Eq.(4.20).

It can be shown (see the Appendix) that the power emitted per unit length of the cylinder in the frequency range $\omega$ and $\omega + d\omega$ can be expressed in the following manner:

$$P_\omega = \frac{4c}{k}\sum_{-\infty}^{+\infty}\int_{-k}^{+k} dh \int_{-\infty}^{+\infty} dh_1\, \lambda_0^2\, [\, |\overline{P}|^2 + |\overline{Q}|^2\,] \qquad (5.13)$$

Essentially the problem reduces to the evaluation of the space average $|\overline{P}|^2$ and $|\overline{Q}|^2$ in terms of the space correlation function Eq.(5.12); this is carried out in detail in the Appendix. The final result can be expressed in the following way:

$$P_\omega = \frac{4\omega(\varepsilon^* - \varepsilon) I_{o\omega}}{i k^2 a^2} \sum_{n=-\infty}^{+\infty} \int_{-k}^{+k} \frac{\lambda_o^2 \, dh}{|\lambda|^4 |\Delta|^2} \left\{ \frac{a}{\lambda^2 - \lambda^{*2}} \Big[ k^2 \mu \left( |\Delta_2|^2 + |\delta|^2 \right) \right.$$

$$\times \left( \lambda^2 J J'^* - \lambda^{*2} J^* J' \right) + \frac{|\Delta_1|^2 + |\delta|^2}{\mu} \Big( h^2 \left( \lambda^2 J J'^* - \lambda^{*2} J^* J' \right)$$

$$\left. + |\lambda|^4 \left( J J'^* - J^* J' \right) \Big) \Big] - k h n J J^* \left[ (\Delta_2^* + \Delta_1^*) \delta + (\Delta_2 + \Delta_1) \delta^* \right] \right\} \quad (5.14)$$

where:

$$\Delta = \Delta_1 \Delta_2 - \delta^2$$

$$\Delta_1 = H' J - \frac{i q \lambda_o^2}{\gamma k \lambda^2} H J'$$

$$\Delta_2 = H' J - \frac{\gamma q \lambda_o^2}{i k \lambda^2} H J'$$

$$\delta = \frac{hn}{ka} \left( 1 - \frac{\lambda_o^2}{\lambda^2} \right) H J$$

$$J = J_n(\lambda a)$$

$$H = H_n^{(2)}(\lambda_o a)$$

with

$$\lambda^2 = q^2 - h^2 \quad ; \quad \lambda_o^2 = k^2 - h^2 \quad ; \quad q = k\sqrt{\varepsilon\mu} \quad ; \quad \gamma = i\sqrt{\frac{\varepsilon}{\mu}}$$

In a plasma the permeability $\mu$ may be assumed equal to unity but the dielectric constant $\varepsilon$ is usually complex and given by[10]

$$\varepsilon = \varepsilon' - i\varepsilon''$$

where

$$\varepsilon' = 1 - \left(\frac{\omega_p}{\omega}\right)^2 \frac{1}{1 + (\nu_m/\omega)^2}$$

$$\varepsilon'' = \left(\frac{\omega_p}{\omega}\right) \frac{(\nu_m/\omega)}{1 + (\nu_m/\omega)^2}$$

$\nu_m$ is the collision frequency for the electrons,

and

$\omega_p$ is the plasma natural frequency given by the relation $\omega_p^2 = \frac{4\pi n e^2}{m}$

where $n$ is the electron concentration ($cm^{-3}$)

$e$ is the electronic charge

$m$ is the mass of the electron

Equation (5.14) constitutes the formal solution (in integral form) to the problem of equilibrium radiation from a cylindrical body. It is valid for any ratio ($a/\lambda$) of cylinder radius to free-space wavelength. It is also valid for any value of $(\omega_p/\omega)^2$ , i.e. for any value of electron concentration and frequency of radiation. However, because of the complexity of the expression, exact numerical solutions may presumably be obtained only with the help of electronic computers. However, in some

limiting cases, it is possible (see Appendix) to simplify equation (5.14) and integrate it directly. Some of these cases will now be discussed.

$(\omega_p/\omega)^2 >> 1$ and $(\nu/\omega) > 1$

This situation represents the low frequency approximation. In this case $\varepsilon'' >> \varepsilon'$ and the plasma essentially behaves as a conductor. Under this approximation Eq.(5.14) reduces to

$$P_\omega = \frac{4(\varepsilon^* - \varepsilon) I_0 \omega}{i k^3 a |\varepsilon|} \sum_{n=-\infty}^{+\infty} \int_{-k}^{+k} \frac{dh}{|H|^2 \, |\rho\sigma - (hn/ka)^2|^2} \times \left\{ \frac{k\sqrt{\mu}(\sqrt{\varepsilon} + \sqrt{\varepsilon^*})}{i(\varepsilon - \varepsilon^*)} \right.$$

$$\left. \times \left[ |\sigma|^2 + \left(\frac{hn}{ka}\right)^2 + \frac{|\varepsilon|}{\mu}\left(|\rho|^2 + \left(\frac{hn}{ka}\right)^2\right)\right] - \left(\frac{hn}{ka}\right)^2 \left(\frac{\rho + \rho^* + \sigma + \sigma^*}{|\varepsilon|\,\mu}\right)\right\} \tag{5.15}$$

with

$$\rho = \frac{H'}{H} + \frac{\mu\lambda_2^2}{i\lambda} \qquad\qquad \sigma = \frac{H'}{H} + \frac{\varepsilon\lambda_2^2}{i\lambda}$$

This solution is valid for any values of $ka$. Although it represents a considerable simplification of Eq.(5.14), it is still too complicated for direct integration. The following approximations are thus studied.

(1) If $ka >> 1$, i.e., if the radius of the cylinder is large Eq.(5.15) can be integrated, it becomes

$$P_{s\omega} = \frac{P_\omega}{2\pi a} = \frac{2KT}{3\pi^2} k^2 \mu d \tag{5.16}$$

where $P_{s\omega}$ is the emissive power per unit area of the cylinder and

$I_{o\omega}$ has been replaced by the Rayleigh-Jeans law,

$d$ is the skin depth.

In terms of the plasma parameters, the skin depth for $(\omega_p/\omega)^2 >> 1$ and $(\nu_m/\omega)^2 > 1$ is given by

$$d = \frac{c}{\omega_p}\sqrt{\frac{2\nu_m}{\omega}}$$

As one would expect the intensity of radiation from a conducting cylinder is proportional to its skin depth.

(2) If $ka << 1$, but $a > d$, Eq.(5.15) becomes after integration

$$P_{s\omega} = \frac{KTk\mu d}{8\pi^2 a^2 \left|\log\frac{ka}{2}\right|^2} \cdot \left|\log\frac{\mu d/a}{\left|\log\frac{ka}{2}\right|}\right| \tag{5.17}$$

This differs from Rytov's result which is

$$P_{s\omega} = \frac{KTk}{4\pi^2 a}\sqrt{\frac{\mu d}{a\left|\log\frac{ka}{2}\right|^2}} \quad .$$

The difference is due to the evaluation of the integral

$$\int_0^{\pi/2} \frac{\cos^3\theta \, d\theta}{\frac{k^2\mu^2d^2}{2} - k\mu d\alpha \cos^2\theta + s^2\cos^4\theta}$$

(3) ka << 1 and a << d

Under this condition the plasma behaves as a poorly conducting cylinder. The solution thus corresponds to the limiting case where $\varepsilon''$ becomes small. It is found that under this approximation Eqn.(5.15) reduces to

$$P_{s\omega} = \frac{2\pi}{3} k a I_{o\omega} \varepsilon'' \tag{5.18}$$

Substituting for $I_{o\omega}$ the Rayleigh-Jean's law one finds that

$$P_{s\omega} = \frac{(KT)\, k^3 a\, \varepsilon''}{6\pi^2}$$

# - APPENDIX -

## DERIVATION OF THE SOLUTION FOR THE EQUILIBRIUM RADIATION FROM A CYLINDER

This appendix is presented to assist the reader in following rapidly the derivation of the solution as first presented by Rytov. In verifying the solution it was found that a considerable amount of algebraic manipulations had to be performed to link some of the equations presented in the Appendix of Rytov's monograph. As a result this section can be regarded as an expansion of the Appendix IV in Rytov's monograph. The notation is kept the same so that it is possible to compare equations directly. It should be noted that one of the approximate solutions derived here does not agree with Rytov's work.

## A - EXPRESSION FOR THE EMITTED POWER IN TERMS OF THE COEFFICIENTS $P_n$ AND $Q_n$

The power emitted per unit length of the cylinder is evaluated from the r-component of the Poynting in the following way (using Eqn. 4.16)

$$P_\omega = r \int_{-\pi}^{+\pi} \overline{S}_{\omega r}\, d\phi = \frac{Cr}{4\pi} \int_{-\pi}^{+\pi} \left[ \overline{E_\phi H_z^*} - \overline{E_z H_\phi^*} + \overline{E_\phi^* H_z} - \overline{E_z^* H_\phi} \right] d\phi \qquad \text{(A-1)}$$

Using Eqns.(5.7a),(5.7b) and (5.1) let us write the components $E_\phi$ and $H_z^*$ ; they are

$$E_\phi = \sum_n \int_h dh \left[ -Z'P - \frac{h\, n\, Z}{qr} Q \right] e^{i(n\phi+hz)}$$

$$H_z^* = -i \sum_{n_1} \int_{h_1} dh_1 \left[ \frac{\lambda^{*2}\, Z^*}{q^*} P^* \right] e^{-i(n_1\phi+h_1 z)}.$$

Their product is given by

$$E_\phi H_z^* = \sum_{n_1} \sum_{n_1} \int_h \int_{h_1} dh\, dh_1 \left\{ i|P|^2 \frac{\lambda^{*2}}{q^*} Z^* Z' + \right.$$

$$\left. + \frac{iQ\, P^*\, \lambda^{*2}\, h\, n}{|q|^2\, r} Z^* Z \right] e^{i[(n-n_1)\phi+(h-h_1)z]} \qquad \text{(A-2a)}$$

In a similar way the other terms of Eqn.(A-1) obtained and are given by

$$E_z H_\phi^* = \sum_n \sum_{n_1} \int_h \int_{h_1} dh\, dh_1 \left[ \frac{i\, Q\, P^*\, \lambda^2 h_1 n_1}{|q|^2\, r} Z^* Z + \right.$$

$$\left. + \; i |Q|^2 \frac{\lambda^2}{q} Z Z'^* \right] e^{i[(n-n_1)\phi + (h-h_1)z]} \qquad \text{(A-2b)}$$

$$E_\phi^* H_z = \sum_n \sum_{n_1} \int_h \int_{h_1} dh\, dh_1 \left[ -i |P|^2 \frac{\lambda^2}{q} Z Z'^* - \right.$$

$$\left. - \frac{i\, P\, Q^*\, \lambda^2 h\, n}{|q|^2\, r} Z^* Z \right] e^{-i[(n-n_1)\phi + (h-h_1)z]} \qquad \text{(A-2c)}$$

$$E_z^* H_\phi = \sum_n \sum_{n_1} \int_h \int_{h_1} dh\, dh_1 \left[ - \frac{i\, P\, Q^*\, \lambda^{*2} h_1 n_1}{|q|^2\, r} Z^* Z - \right.$$

$$\left. - \; i |Q|^2 \frac{\lambda^{*2}}{q^*} Z^* Z' \right] e^{-i[(n-n_1)\phi + (h-h_1)z]} \quad . \qquad \text{(A-2d)}$$

In combining the equations (A-2) with (A-1) one makes use of

$$\int_{-\pi}^{+\pi} e^{i(n-n_1)\phi}\, d\phi = 2\pi\, \delta_{nn_1} \quad .$$

Also assume $h = h_1$. Since, as will be shown later the coefficients $|\overline{P}|^2$ and $|\overline{Q}|^2$ contain a Dirac delta function, one obtains

$$P_\omega = \frac{iCr}{2k} \sum_n \int_h \int_{h_1} dh\, dh_1 \left[ |\overline{P}|^2 + |\overline{Q}|^2 \right] \lambda_e^2 \, (Z^* Z' - Z Z'^*) \qquad \text{(A-3)}$$

(R-IV.13)†

† Equations preceded by the letter R refer to the equations in Rytov's monograph.[1]

where $\lambda^2 = \lambda_0^2 = q^2 - h^2 = k^2 - h^2$ is real.

But here $Z = H_n^{(2)}(\lambda_0 r)$, so one has

$$Z^* = [H_n^{(2)}(\lambda_0 r)]^* = H_n^{(1)}(\lambda_0^* r)$$

$$Z' = \frac{d}{dr}[H_n^{(2)}(\lambda_0 r)] = \lambda_0 [H_{n-1}^{(2)}(\lambda_0 r) - \frac{n}{\lambda_0 r} H_n^{(2)}(\lambda_0 r)]$$

$$Z'^* = \left\{\frac{d}{dr}[H_n^{(2)}(\lambda_0 r)]\right\}^* = \lambda_0^* \left\{[H_{n-1}^{(2)}(\lambda_0 r)]^* - \frac{n}{\lambda_0^* r}[H_n^{(2)}(\lambda_0 r)]^*\right\}$$

$$Z'^* = \lambda_0^*[H_{n-1}^{(1)}(\lambda_0^* r) - \frac{n}{\lambda_0^* r} H_n^{(1)}(\lambda_0^* r)]$$

Therefore

$$Z^*Z' - ZZ'^* = \lambda_0 H_{n-1}^{(2)}(\lambda_0 r)\, H_n^{(1)}(\lambda_0^* r) - \lambda_0^* H_{n-1}^{(1)}(\lambda_0^* r)\, H_n^{(2)}(\lambda_0 r) \qquad (A-4)$$

Two cases must be studied.

(a) for $|h| < k$, $\lambda_0$ is real and

$$Z^*Z' - ZZ'^* = \lambda_0[H_{n-1}^{(2)}(\lambda_0 r)\, H_n^{(1)}(\lambda_0 r) - H_{n-1}^{(1)}(\lambda_0 r)\, H_n^{(2)}(\lambda_0 r)]$$

which gives using the formula 129, p. 198 in McLachlan[11]

$$Z^*Z' - ZZ'^* = -\frac{4i}{\pi r} \qquad (A-5)$$

(b) for $|h| > k$ $\lambda_0$ is imaginary

In this case $\lambda_0^* r = -\lambda_0 r$

$$H_n^{(2)}(\lambda_o r) = J_n - iN_n \quad ; \quad H_{n-1}^{(2)}(\lambda_o r) = J_{n-1} - iN_{n-1}$$

$$H_n^{(1)}(-\lambda_o r) = (-1)^{n+1}(J_n - iN_n) \quad ; \quad H_{n-1}^{(1)}(-\lambda_o r) = (-1)^{n}(J_{n-1} - iN_{n-1})$$

Inserting these expressions into Eqn.(A-4) gives

$$Z^*Z' - ZZ'^* = 0 \tag{A-6}$$

It is thus seen that Eqn.(A-6) restricts the range of integration of $h$ from $-k$ to $+k$. With the help of Eqn.(A-5) it is then possible to write Eqn.(A-3) as follows.

$$P_{\omega_r} = \frac{2C}{\pi k} \sum_{n=-\infty}^{+\infty} \int_{-k}^{+k} dh \int_{-\infty}^{+\infty} dh_1 \, \lambda_o^2 \left( |\bar{P}|^2 + |\bar{Q}|^2 \right) \qquad \text{(A-7)} \quad \text{(R-IV.16)}$$

## B - EXPRESSIONS FOR THE COEFFICIENTS $P_n$ AND $Q_n$ IN TERMS OF THE OTHER COEFFICIENTS $A_n$ AND $B_n$

The next step consists in evaluating the coefficients $|\overline{P}|^2$ and $|\overline{Q}|^2$ in terms of the function $G_{nh}(r)$ and the other parameters of the problems. This is done by substituting Eqns.(5.5),(5.6) and (5.7) into the boundary conditions given by Eqn.(5.10). The $\phi$-components of the electric field at boundary r=a is given by, using Eqns.(5.1),

$$E_{c\phi} = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dh \Big( -AJ' - B \frac{hn}{qa} J + iC \frac{n}{a} J - \tilde{A}N' -$$

$$- \tilde{B} \frac{hn}{qa} N + i\tilde{C} \frac{n}{a} N \Big) e^{i(n\phi+hz)} \tag{A-8a}$$

$$E_{r\phi} = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dh \left( -A_r J' - B_r \frac{hn}{qa} J_n \right) e^{i(n\phi+hz)} \tag{A-8b}$$

$$E_{\phi} = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dh \left( -P_n H' - Q_n \frac{hn}{ka} H \right) e^{i(n\phi+hz)} \tag{A-8c}$$

where $J = J_n(\lambda a)$ $\quad J' = \frac{d}{da} [J_n(\lambda a)]$

$N = N_n(\lambda a)$ $\quad N' = \frac{d}{da} [N_n(\lambda a)]$

$H = H_n^{(2)}(\lambda_o a)$ $\quad H' = \frac{d}{da} [H_n^{(2)}(\lambda_o a)]$

Substituting Eqns.(A-8) into the first boundary condition leads to

$$AJ' + \tilde{A}N' + A_rJ' + \frac{hn}{qa}(BJ + \tilde{B}N + B_rJ) = P_nH' + \frac{hn}{ka} Q_nH \qquad \text{(A-9a)}$$

Similarly with the other boundary conditions one obtains

$$\frac{\lambda^2}{q}(BJ + \tilde{B}N + B_rJ) = \frac{\lambda_a^2}{k} Q_nH \qquad \text{(A-9b)}$$

$$\gamma\left[BJ' + \tilde{B}N' + B_rJ' + \frac{hn}{qa}(AJ + \tilde{A}N + A_rJ)\right]$$

$$= i\left(Q_nH' + \frac{hn}{ka} P_nH\right) \qquad \text{(A-9c)}$$

$$\frac{\gamma\lambda^2}{q}(AJ + \tilde{A}N + A_rJ) = \frac{i\lambda_a^2}{k} P_nH \ . \qquad \text{(A-9d)}$$

$P_n$ and $Q_n$ may be expressed in terms of $\tilde{A}$ and $\tilde{B}$ in the following way. Rewriting (A-9b) and (A-9c) as follows

$$(B + B_r)J + \tilde{B}N = \frac{q}{k}\frac{\lambda_a^2}{\lambda^2} Q_nH \qquad \text{(A-9b)}$$

$$(B + B_r)J' + \tilde{B}N + \frac{hn}{qa}(AJ + \tilde{A}N + A_rJ) = \frac{i}{\gamma}\left(Q_nH' + \frac{hn}{ka} P_nH\right) \qquad \text{(A-9c)}$$

Multiplying (A-9b) by J' and (A-9c) by J and subtracting one from the other leads to

$$\tilde{B}(NJ' - N'J) - \frac{hn}{qa}(AJ + \tilde{A}N + A_rJ)J = -\frac{i}{\gamma}\Delta_2 Q_n - \frac{i}{\gamma}\frac{hn}{ka} P_nHJ$$

where

$$\Delta_2 = H'J - \frac{\gamma q \lambda_0^2}{ik\lambda^2} HJ' \qquad \text{(A-10a)}$$

Substituting $(AJ + \tilde{A}N + A_r J)$ from Eqn.(A-9d) into this expression leads to, using the relation $JN' - J'N = \frac{2}{\pi a}$,

$$\delta P_n + \Delta_2 Q_n = \frac{2\gamma}{i\pi a}\tilde{B} \qquad \text{(A-11a)}$$

In a similar way one obtains from the Eqns.(A-9)

$$\Delta_1 P_n + \delta Q_n = \frac{2\tilde{A}}{\pi a} \qquad \text{(A-11b)}$$

where

$$\delta = \frac{hn}{ka}\left(1 - \frac{\lambda_0^2}{\lambda^2}\right) HJ \qquad \text{(A-10b)}$$

and

$$\Delta_1 = H'J - \frac{iq\lambda_0^2}{\gamma k\lambda^2} H J' \qquad \text{(A-10c)}$$

Solving Eqns.(A-11a) and (A-11b) for $P_n$ and $Q_n$ leads to

$$P_n = \frac{2}{\pi a \Delta}(\Delta_2\tilde{A} + i\gamma\delta\tilde{B}) \qquad \text{(A-12a)}$$

$$Q_n = -\frac{2}{\pi a \Delta}(\delta\tilde{A} + i\gamma\Delta_1\tilde{B}) \qquad \text{(A-12b)}$$

where

$$\Delta = \Delta_1\Delta_2 - \delta^2 \qquad \text{(A-10d)}$$

By substituting Eqns.(A-12) into Eqn.(A-7) it is found that the expression for the power emitted per unit length of the cylinder is given by

$$P_\omega = \frac{8C}{\pi^2 k a^2} \sum_{n=-\infty}^{+\infty} \int_{-k}^{+k} dh \int_{-\infty}^{+\infty} dh_1 \frac{\Delta_0^2}{|\Delta|^2} \Big[ |\bar{\tilde{A}}|^2 \left( |\Delta_2|^2 + |\delta|^2 \right)$$

$$+ |\gamma|^2 \, |\bar{\tilde{B}}|^2 \left( |\Delta_1|^2 + |\delta|^2 \right) + \overline{\tilde{A}^* \tilde{B}} \, i \gamma \left( \Delta_2^* \delta + \Delta_1 \delta^* \right)$$

$$- \overline{\tilde{A} \tilde{B}^*} \, i \gamma^* \left( \Delta_2 \delta^* + \Delta_1^* \delta \right) \Big] \qquad \text{(A-13)} \quad \text{(R IV.17)}$$

The problem is now reduced to the evaluation of the coefficients $\tilde{A}_n$ and $\tilde{B}_n$ in terms of the source function $G_{nh}(r)$.

## C - DETERMINATION OF THE COEFFICIENTS $\tilde{A}_n$ AND $\tilde{B}_n$

Inserting the expressions for $\vec{E}_e$, $\vec{H}_e$ and $\vec{K}$, as given by Eqns.(5.5a), (5.5b) and (5.8), into the second Maxwell equation leads to, since the same summation and integration range applies to all terms,

$$y\,\mathrm{Curl}\,(A\vec{N} + B\vec{M} + \tilde{A}\tilde{\vec{N}} + \tilde{B}\tilde{\vec{M}})$$

$$= ik\varepsilon(A\vec{M} + B\vec{N} + C\vec{L} + \tilde{A}\tilde{\vec{M}} + \tilde{B}\tilde{\vec{N}} + \tilde{C}\tilde{\vec{L}} + G_n\, e^{i(n\phi+hz)}) \qquad \text{(A-14)}$$

But from Eqns.(5.3a) and (5.3b) one can write

$$\mathrm{Curl}\,A\vec{N} = A\,\mathrm{Curl}\,\vec{N} - [\nabla A\times\vec{N}] = qA\vec{M} - [\nabla A\times\vec{N}]$$

$$\mathrm{Curl}\,B\vec{M} = B\,\mathrm{Curl}\,\vec{M} - [\nabla B\times\vec{M}] = qB\vec{N} - [\nabla B\times\vec{M}] \qquad \text{(A-15)}$$

$$\mathrm{Curl}\,\tilde{A}\tilde{\vec{N}} = \tilde{A}\,\mathrm{Curl}\,\tilde{\vec{N}} - [\nabla\tilde{A}\times\tilde{\vec{N}}] = q\tilde{A}\tilde{\vec{M}} - [\nabla\tilde{A}\times\tilde{\vec{N}}]$$

$$\mathrm{Curl}\,\tilde{B}\tilde{\vec{M}} = \tilde{B}\,\mathrm{Curl}\,\tilde{\vec{M}} - [\nabla\tilde{B}\times\tilde{\vec{M}}] = q\tilde{B}\tilde{\vec{N}} - [\nabla\tilde{B}\times\tilde{\vec{M}}]$$

Introducing Eqns.(A-15) into Eqn.(A-14) gives

$$\nabla A\times\vec{N} + \nabla B\times\vec{M} + \nabla\tilde{A}\times\tilde{\vec{N}} + \nabla\tilde{B}\times\tilde{\vec{M}} = q[C\vec{L} + \tilde{C}\tilde{\vec{L}} + G_n\, e^{i(n\phi+hz)}] \qquad \text{(A-16a)}$$

Proceeding in the same way with the first equation in Eqns.(4.1), one obtains

$$\nabla A\times\vec{M} + \nabla B\times\vec{N} + \nabla C\times\vec{L} + \nabla\tilde{A}\times\tilde{\vec{M}} + \nabla\tilde{B}\times\tilde{\vec{N}} + \nabla\tilde{C}\times\tilde{\vec{L}} = 0 \qquad \text{(A-16b)}$$

The gradients of $A$, $B$, $C$, $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ have a component only with respect to $r$. So

$$\nabla A = A'\vec{i}_1 \quad ; \quad \nabla B = B'\vec{i}_1 \quad ; \ldots \nabla\tilde{C} = \tilde{C}'i_1$$

where $$A' = \frac{dA}{dr} \quad .$$

With the help of the Eqns.(5.1) it is then possible to write the three components of Eqn.(A-16a). One obtains for the component of $\vec{i}_1$:

$$CZ' + \widetilde{\tilde{C}Z'} + G_{nr} = 0$$

for $\vec{i}_2$:

$$\lambda^2(A'Z + \tilde{A}'\tilde{Z}) + \frac{i\,n\,q^2}{r}(CZ + \widetilde{\tilde{C}Z}) = -q^2\, G_{n\phi}$$

for $\vec{i}_3$:

$$\frac{hn}{qr}(A'Z + \widetilde{\tilde{A}Z}) + (B'Z' + \tilde{B}'\tilde{Z}') + i\,q\,h(CZ + \widetilde{\tilde{C}Z}) = -q\,G_{nz}$$

where the factor $e^{i(n\phi+hz)}$ which appears in each term has been dropped. Using the fact that

$$Z = J_n(\lambda_r) = J \quad ; \quad Z' = J' \qquad \text{(Bessel Function)}$$

$$\tilde{Z} = N_n(\lambda_r) = N \quad ; \quad \tilde{Z}' = N' \qquad \text{(Neumann Function)}$$

these equations become

$$CJ' + \tilde{C}N' = -G_{nr}$$

$$\lambda^2(A'J + \tilde{A}'N') + \frac{i\,n\,q^2}{r}(CJ + \tilde{C}N) = -q^2 G_{n\phi} \qquad \text{(A-17)}$$

$$\frac{hn}{qr}(A'J + \tilde{A}'N) + (B'J' + \tilde{B}'N') + i\,q\,h(CJ + \tilde{C}N) = -q\,G_{nz}$$

Carrying out a similar operation with Eqn.(A-3b) one gets:

$$\lambda^2(B'J + \tilde{B}'N) - i\,h\,q(C'J + \tilde{C}'N) = 0$$

$$(A'J + A'N') + \frac{hn}{qr}(B'J + B'N) - \frac{in}{r}(C'J + C'N) = 0 \qquad \text{(A-18)}$$

There are five equations and six unknown; as a supplementary condition Rytov chooses

$$CJ + \tilde{C}N = 0 \qquad \text{(A-19)}$$

This condition essentially implies that the $\phi(\vec{i}_2)$ and $z(\vec{i}_3)$ components in the term $\vec{CL} + \tilde{\vec{CL}}$ of Eqn.(A-16a) are zero. Differentiating (A-19) with respect to r and comparing with the first equation in (A-17) leads to

$$C'J + \tilde{C}'N = G_{nr} \qquad \text{(A-20)}$$

Substituting Eqns.(A-19) and (A-20) into the last two equations of (A-17) and the equations in (A-18) gives

$$A'J + \tilde{A}'N = -\frac{q^2}{\lambda^2}G_{n\phi} \qquad \text{(A-21a)}$$

$$\frac{hn}{qr}(A'J + \tilde{A}'N) + (B'J' + \tilde{B}'N') = -q\,G_{nz} \qquad \text{(A-21b)}$$

$$B'J + \tilde{B}'N = -i\frac{hq}{\lambda^2}G_{nr} \qquad \text{(A-21c)}$$

$$(A'J' + \tilde{A}'N') + \frac{hn}{qr}(B'J + \tilde{B}'N) = \frac{in}{r}G_{nr} \qquad \text{(A-21d)}$$

By substitution, it is then possible to isolate the coefficients $A'$, $\tilde{A}'$, $B'$ and $\tilde{B}'$. For example $B'$ is obtained as follows: introducing (A-21a) into (A-21b) gives

$$B'J' + \tilde{B}'N' = \frac{hnq}{\lambda^2 r} G_{n\phi} - qG_{ns} \tag{A-22}$$

Multiplying (A-22) by $N$ and (A-21c) by $N'$ and subtracting leads to

$$B' = -\frac{\pi r q}{2}\left[\frac{ih}{\lambda^2} G_{nr} N' + \left(G_{ns} - \frac{hn}{\lambda^2 r} G_{n\phi}\right) N\right] \tag{A-23a}$$

where the relation (McLachlan, p. 197)[11]

$$-J'N + JN' = \frac{2}{\pi r}$$

has been used. Similarly one obtains

$$A' = -\frac{\pi q^2 r}{2\lambda^2}\left(G_{n\phi} N' + \frac{in}{r} G_{nr} N\right) \tag{A-23b}$$

$$\tilde{B}' = \frac{\pi q r}{2}\left[\frac{ih}{\lambda^2} G_{nr} J - \left(G_{nr} - \frac{hn}{\lambda^2 r} G_{n\phi}\right) J\right] \tag{A-23c}$$

$$\tilde{A}' = \frac{\pi q^2 r}{2\lambda^2}\left(G_{n\phi} J' + \frac{in}{r} G_{nr} J\right) \tag{A-23d}$$

(R-IV.6)

From equations (A-17a) and (A-19) $C$ and $\tilde{C}$ may be isolated and they are given by:

$$C = \frac{\pi r}{2} G_{nr} N \tag{A-23e}$$

$$\tilde{C} = -\frac{\pi r}{2} G_{nr} J \tag{A-23f}$$

$A$, $B$, $\tilde{A}$ and $\tilde{B}$ can simply be obtained from (A-23a,b,c,d) by integration:

$$A = \int_0^a A'\, dr \quad ; \quad B = \int_0^a B'\, dr \quad \text{etc.}$$

The equations (A-23) thus describe the coefficients $A$, $B$, $C$, $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ in terms of the fluctuation function $G_{nz}$.

It is now possible to evaluate the coefficients $\overline{|\tilde{A}|^2}$, $\overline{|\tilde{B}|^2}$, $\overline{\tilde{A}^*\tilde{B}}$ and $\overline{\tilde{A}\tilde{B}^*}$ which appear in Eqn.(A-13). Let us write, from (A-23c) and (A-23d), the $\tilde{A}$ and $\tilde{B}$ coefficient

$$\tilde{A} = \frac{\pi q^2}{2\lambda^2} \int_0^a \left[ G_{n\phi}\, J' + \frac{in}{r}\, G_{nr}\, J \right] r\, dr \tag{A-24a}$$

$$\tilde{B} = \frac{\pi q}{2} \int_0^a \left\{ \frac{ih}{\lambda^2}\, G_{n_1 r_1}\, J'_{n_1}(\lambda r_1) - \left[ G_{n_1 z} - \frac{hn_1}{\lambda^2 r_1}\, G_{n_1\phi} \right] J_{n_1}(\lambda r_1) \right] r_1\, dr_1$$

(A-24b)

so

$$\tilde{A}^* = \frac{\pi q^{*2}}{2\lambda^{*2}} \int_0^a \left[ G^*_{n_1\phi}\, J'^* - \frac{in_1}{r_1}\, G^*_{n_1 r_1}\, J^* \right] r_1\, dr_1 \tag{A-24c}$$

Multiplying (A-24a) by (A-24b) and taking the average leads to

$$\overline{|\tilde{A}|^2} = \frac{\pi^2 |q^2|\, |q^*|^2}{4\lambda^{*2}\lambda^2} \int_0^a\!\!\int_0^a \Big[ \overline{G_{n\phi}\, G^*_{n_1\phi}}\, J'(\lambda r)\, J'^*(\lambda r_1) +$$

$$+ \frac{n^2}{rr_1}\, \overline{G_{nr}\, G^*_{n_1 r_1}}\, J_n(\lambda r)\, J^*_{n_1}(\lambda r_1) - \frac{in_1}{r_1}\, \overline{G_{n\phi}\, G^*_{n_1 r_1}}\, J'_n(\lambda r)\, J^*_{n_1}(\lambda r_1) +$$

$$+ \frac{in}{r}\, \overline{G_{nr}\, G^*_{n_1\phi}}\, J_n(\lambda r)\, J'^*_{n_1}(\lambda r_1) \Big]\, r dr\, r_1 dr_1 \tag{A-25}$$

Using the relation (5.12)

$$\overline{G_{nh\alpha}(r)\, G_{n_1h_1\beta}(r_1)} = \frac{|\varepsilon-1|^2}{4\pi^2|\varepsilon|^2}\, C\, \delta_{nn_1}\, \delta_{\alpha\beta}\, \frac{\delta(r-r_1)}{r}\, \delta(h-h_1)$$

where $\alpha,\beta = r, \phi, z$

One can make $n = n_1$ and $\alpha = \beta$ and Eqn.(A-25) reduces to

$$\overline{|\bar{A}|^2} = \frac{|c|^2\,|c^*|^2}{16\lambda^{*2}\lambda^2}\,\frac{|\varepsilon-1|^2}{|\varepsilon|^2}\, C\, \delta(h-h_1) \int_0^a\!\!\int_0^a \Big[ J_n'(\lambda r)\, J_n'^*(\lambda r_1)\, \frac{\delta(r-r_1)}{r} +$$

$$+ \frac{n^2}{rr_1}\,\frac{\delta(r-r_1)}{r}\, J_n(\lambda r)\, J_n^*(\lambda r_1)\Big]\, r dr\, r_1 dr_1 \qquad \text{(A-26)}$$

The integral in (A-26) can then be written as

$$I = \int_0^a J_n'(\lambda^* r_1) r_1 dr_1 \int_0^a J_n'(\lambda r)\, \delta(r-r_1) dr +$$

$$+ \int_0^a \frac{n^2}{r_1} J_n(\lambda^* r_1) r_1 dr_1 \int_0^a \frac{J_n(\lambda r)}{r}\, \delta(r-r_1) dr$$

which then becomes, using the properties of the Dirac delta function

$$I = \int_0^a J_n'(\lambda^* r_1)\, J_n'(\lambda r_1) r_1 dr_1 + \int_0^a \frac{n^2 J_n(\lambda^* r_1)\, J_n(\lambda r_1)}{r_1}\, dr_1$$

Integrating by parts the first integral leads to

$$I = aJ_n'(\lambda a)\, J_n(\lambda^* a) - \int_0^a J_n(\lambda^* r)\,[rJ_n''(\lambda r) + J_n'(\lambda r)\,]\,dr$$

$$+ \int_0^a n^2 \frac{J_n(\lambda^* r_1)\, J_n(\lambda r_1)}{r_1}\, dr_1$$

Since the Bessel equation can be written as

$$rJ_n''(\lambda r) + J_n'(\lambda r) = -\frac{\lambda[(\lambda r)^2 - n^2]}{(\lambda r)}\, J_n(\lambda r)$$

the integral can then be reduced to

$$I = aJ_n'(\lambda a)\, J_n(\lambda^* a) + \lambda^2 \int_0^a J_n(\lambda^* r)\, J_n(\lambda r)\, r\, dr$$

From the formula no.(79) in McLachlan,[11] one can write

$$\int_0^a J_n(\lambda^* r)\, J_n(\lambda r)\, r\, dr = \frac{a}{\lambda^2 - \lambda^{*2}}\Big[\, J_n(\lambda a)\, J_n'(\lambda^* a) - J_n(\lambda^* a)\, J_n'(\lambda a)\Big].$$

The integral then becomes

$$I = \frac{a}{\lambda^2 - \lambda^{*2}}\Big[\, \lambda^2 J_n(\lambda a)\, J_n'(\lambda^* a) - \lambda^{*2} J_n(\lambda^* a)\, J_n'(\lambda a)\Big].$$

This expression can then be substituted into Eqn.(A-26) and the following expression is obtained for $\overline{|\tilde{X}|^2}$

$$\overline{|\tilde{X}|^2} = \frac{|q|^2 |q^*|^2 |\varepsilon - 1|^2\, C}{16|\varepsilon|^2 |\lambda|^4}\; \frac{(\lambda^2 JJ'^* - \lambda^{*2} J^* J')\, a}{\lambda^2 - \lambda^{*2}}\; \delta(h - h_1) \qquad \text{(A-27a)}$$

(R-IV.18)

In a similar way one obtains the following coefficients.

$$\overline{|\mathfrak{B}|^2} = \frac{|q|^2|\varepsilon-1|^2C}{16|\varepsilon|^2|\lambda|^4} \cdot \frac{h^2(\lambda^2JJ'^*-\lambda^{*2}J^*J')+|\lambda|^4(JJ'^*-J^*J')}{\lambda^2-\lambda^{*2}}\,\delta(h-h_1)$$

(A-27b)

$$\overline{\mathfrak{A}\mathfrak{B}^*} = \frac{|q|^2|\varepsilon-1|^2C}{16|\varepsilon|^2|\lambda|^4} \cdot q\,n\,h\,J\,J^*\,\delta(h-h_1) \qquad \text{(A-27c)}$$

$$\overline{\mathfrak{A}^*\mathfrak{B}} = \frac{|q|^2|\varepsilon-1|^2C}{16|\varepsilon|^2|\lambda|^4}\, q^*n\,h\,J\,J^*\,\delta(h-h_1) \qquad \text{(A-27d)}$$

It is now possible to write the general solution for the equilibrium radiation from a cylinder by substituting the Eqns.(A-27) into (A-13) and recalling that C is given by Eqn.(4.20) and

$$\gamma = i\sqrt{\frac{\varepsilon}{\mu}} \quad ; \quad \gamma^* = -i\sqrt{\frac{\varepsilon^*}{\mu}} \quad ; \quad |\gamma|^2 = \frac{|\varepsilon|}{\mu}$$

$$i\gamma\, q^* = -i\gamma^* q = -k|\varepsilon|$$

One then obtains

$$P_\omega = \frac{4\mu(\varepsilon^*-\varepsilon)I_{o\omega}}{ik^2a^2} \sum_{n=-\infty}^{+\infty} \int_{-k}^{+k} \frac{\lambda_0^2\,dh}{|\lambda|^4|\Delta|^2} \Bigg\{ \frac{2}{\lambda^2-\lambda^{*2}} \Bigg[ k^2\mu\,(|\Delta_2|^2+|\delta|^2)$$

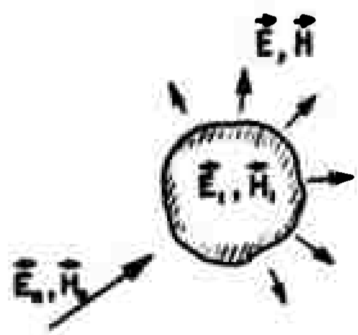
$$\times(\lambda^2JJ'^*-\lambda^{*2}J^*J') + \frac{|\Delta_1|^2+|\delta|^2}{\mu}\Big(h^2(\lambda^2JJ'^*-\lambda^{*2}J^*J')$$

$$+|\lambda|^4(JJ'^*-J^*J')\Big)\Bigg] - k\,h\,n\,J\,J^*\Big[(\Delta_2^*+\Delta_1^*)\delta + (\Delta_2+\Delta_1)\delta^*\Big]\Bigg\}$$

(A-29)
(5-14)
(R-IV.19)

D - THE ABSORPTIVITY OF THE CYLINDER

It will be shown here that in contrast with the above development of the solution for the equilibrium radiation from a cylinder a simpler approach may be taken if one computes the power absorptivity of the cylinder to a test wave incident upon it. This method is particularly interesting since it allows one to treat any problem of equilibrium radiation as a scattering problem of e-m waves by any given body. The problem treated here also constitutes a unique test to the general formulation of Kirchhoff's law of radiation as formulated by Rytov[1] and Levin[12].




The e-m field scattered by a body.

Consider a test-wave defined by the fields $\vec{E}_o$ and $\vec{H}_o$ incident upon the body and compute the flux of energy scattered from the body and expressed in terms of the fields $\vec{E}$ and $\vec{H}$. The field inside the body due to absorption and transmission of the waves is represented by $\vec{E}_1$ and $\vec{H}_1$.

For the case of an infinitely long cylinder of radius a, the incident test wave is expressed by cylindrical waves travelling towards the cylinder. The field of these waves must satisfy Maxwell's equations so in accordance with Eqn.(5.4) one can write a

particular solution for the incident field as

$$\vec{E}_0 = P_0 \vec{M}^* + Q_0 \vec{N}^* \tag{A-30a}$$

$$\vec{H}_0 = i(P_0 \vec{N}^* + Q_0 \vec{M}^*) \tag{A-30b}$$

where $\vec{M}^*$ and $\vec{N}^*$ contain cylindrical function of the type $Z_n = H_n^{(1)}(\lambda_0 r)$ (Hankel functions of the first kind) or according to our previous notation

$$Z_n = H_n^{(1)}(\lambda_0 r) = H_n^{(2)*}(\lambda_0 r) = H^*$$

The field inside the cylinder can be described in the same way as the field $\vec{E}_r$ in Eqn.(5.4), so a particular solution is given by

$$\vec{E}_1 = A_1\vec{M} + B_1\vec{N} \tag{A-31a}$$

$$\vec{H}_1 = y(A_1\vec{N} + B_1\vec{M}) \tag{A-31b}$$

where $\vec{M}$ and $\vec{N}$ contain cylindrical functions of the type

$$Z_n = J_n(\lambda r)$$

The scattered field is then represented by outgoing cylindrical waves and can be expressed by

$$\vec{E} = P\vec{M}^* + Q\vec{N}^* \tag{A-32a}$$

$$\vec{H} = i(P\vec{N}^* + Q\vec{M}^*) \tag{A-32b}$$

where the cylindrical functions in $\vec{M}^*$ and $\vec{N}^*$ are represented by $Z_n = H_n^{(2)}(\lambda_0 r) = H$ (Hankel function of the second kind).

Since at the boundary of the cylinder the tangential components of the field must be equal the boundary conditions are then given by

$$\begin{aligned} E_{1\phi} &= E_{o\phi} + E_{\phi} \quad ; \quad H_{1\phi} = H_{o\phi} + H_{\phi} \\ E_{1z} &= E_{oz} + E_z \quad ; \quad H_{1z} = H_{oz} + H_z \end{aligned} \qquad \text{at } r = a \qquad \text{(A-33)}$$

It is possible to derive four equations relating the various coefficients of the waves by introducing Eqns.(A-30),(A-31) and (A-32) and using the relations (5.1). For example let us write the $\phi$-component of the electric fields.

$$E_{1\phi} = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dh \left( -A_1 J' - \frac{hn}{qa} B_1 J \right) e^{i(n\phi + hz)} \qquad \text{(A-34a)}$$

$$E_{o\phi} = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dh \left( -P_o H^{*\prime} - \frac{hn}{ka} Q_o H^{*} \right) e^{i(n\phi + hz)} \qquad \text{(A-34b)}$$

$$E_{\phi} = \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dh \left( -PH' - \frac{hn}{ka} QH \right) e^{i(n\phi + hz)} \qquad \text{(A-34c)}$$

When substituted into Eqn.(A-33) the equations (A-34) lead to

$$A_1 J' + \frac{hn}{qa} B_1 J = P_o H'^{*} + \frac{hn}{ka} Q_o H^{*} + PH' + \frac{hn}{ka} QH \qquad \text{(A-35a)}$$

Similarly by matching the other components of the E and H fields one gets

$$\frac{\lambda^2}{q} B_1 J = \frac{\lambda_2^2}{k} (Q_0 H^* + QH) \tag{A-35b}$$

$$y\left(B_1 J' + \frac{hn}{qa} A_1 J\right) = i\left(Q_0 H'^* + \frac{hn}{ka} P_0 H^* + QH' + \frac{hn}{ka} PH\right) \tag{A-35c}$$

$$\frac{y\lambda^2}{q} A_1 J = \frac{i\lambda_2^2}{k} (P_0 H^* + PH) \tag{A-35d}$$

Substituting $A_1$ from (A-35d) and $B_1$ from (A-35b) into (A-35a) and (A-35c) leads to the following relations between P, Q and $P_0$, $Q_0$

$$\Delta_1 P + \delta Q = -(\phi_1 P_0 + \phi Q_0) \tag{A-36a}$$

$$\delta P + \Delta_2 Q = -(\phi P_0 + \phi_2 Q_0) \tag{A-36b}$$

where $\Delta_1$, $\Delta_2$ and $\delta$ are given by (A-10a), (A-10b) and (A-10c) and

$$\phi_1 = H'^* J - \frac{iq\lambda_2^2}{yk\lambda^2} H^* J' \tag{A-37a}$$

$$\phi_2 = H'^* J - \frac{yq\lambda_2^2}{ik\lambda^2} H^* J' \tag{A-37b}$$

$$\phi = \frac{hn}{ka}\left(1 - \frac{\lambda_2^2}{\lambda^2}\right) H^* J \tag{A-37c}$$

The expressions for P and Q can then be obtained by solving the equations (A-36); one gets

$$P = \frac{1}{\Delta}\left[(\phi\delta - \phi_1\Delta_2)P_0 + (\phi_2\delta - \phi\Delta_2)Q_0\right] \tag{A-38a}$$

(R-IV.25)

$$Q = \frac{1}{\Delta}\left[(\phi_1\delta - \phi\Delta_1)P_0 + (\phi\delta - \phi_2\Delta_1)Q_0\right] \tag{A-38b}$$

It is seen that the scattered P-waves as well as the Q-waves depends on both the $P_0$ and $Q_0$ incident waves. This is the result of absorption of energy by the body. It can be shown that if the cylinder were a perfect conductor the P-waves would only depend on the $P_0$-waves and the Q-waves on the $Q_0$-waves. It is thus convenient to define an absorption coefficient for each type of incident waves. If a $P_0$-wave is incident on the cylinder then the absorptivity can be written as

$$A_{Pn}(h) = 1 - \frac{\left[|P|^2 + |Q|^2\right]_{Q_0=0}}{|P_0|^2} \qquad \text{(A-39a)}$$

This expression simply describes the absorptivity in terms of the fraction of the reflected power. Similarly for a $Q_0$-wave, one writes

$$A_{Qn}(h) = 1 - \frac{\left[|P|^2 + |Q|^2\right]_{P_0=0}}{|Q_0|^2} \qquad \text{(A-39b)}$$

In order to evaluate $A_{Pn}(h)$ let us first compute $|P|^2$ and $|Q|^2$ for $Q_0 = 0$. Eqn.(A-38a) then becomes

$$P = \frac{1}{\Delta}\left[(\phi\delta - \phi_1\Delta_2)\right]P_0$$

$$P^* = \frac{1}{\Delta^*}\left[(\phi^*\delta^* - \phi_1^*\Delta_2^*)\right]P_0^*$$

so

$$|P|^2 = \frac{1}{|\Delta|^2}\,|\phi\delta - \phi_1\Delta_2|^2\,|P_0|^2 \qquad \text{(A-40a)}$$

Similarly

$$|Q|^2 = \frac{1}{|\Delta|^2} \; |\phi_1\delta - \phi\Delta_1|^2 \; |P_0|^2 \qquad \text{(A-40b)}$$

Substituting Eqns.(A-40a) and (A-40b) into (A-39a), one gets

$$A_{Pn}(h) = 1 - \frac{1}{|\Delta|^2}\left[|\phi\delta - \phi_1\Delta_2|^2 + |\phi_1\delta - \phi\Delta_1|^2\right]$$

and recalling that from Eqn.(A-10d)

$$|\Delta|^2 = |\Delta_1\Delta_2 - \delta^2|^2$$

$A_{Pn}(h)$ then becomes

$$A_{Pn}(h) = \frac{1}{|\Delta|^2}\left[|\Delta_1\Delta_2 - \delta^2|^2 - |\phi\delta - \phi_1\Delta_2|^2 - |\phi_1\delta - \phi\Delta_1|^2\right]. \qquad \text{(A-41)}$$

After expanding each term in Eqn.(A-41), adding and subtracting $|\delta|^2|\Delta_1|^2$ and noting that $|\phi|^2 = |\delta|^2$(from A-10b and A-37c) the expression for $A_{Pn}(h)$ becomes

$$A_{Pn}(h) = \frac{1}{|\Delta|^2}\Big\{(|\Delta_2|^2 + |\delta|^2)(|\Delta_1|^2 - |\phi_1|^2) +$$

$$+ (\phi_1\phi^* - \delta^*\Delta_1)(\delta^*\Delta_2 + \delta\Delta_1^*) + (\phi_1^*\phi - \delta\Delta_1^*)(\delta\Delta_2^* + \delta^*\Delta_1)\Big\} \qquad \text{(A-42a)}$$

In order to arrive at the final form for $A_{Pn}(h)$ three relations must be proved.

To show that

$$|\Delta_1|^2 - |\phi_1|^2 = \frac{4i\lambda_0^2}{\pi a|\lambda|^4}\left(\lambda^2 JJ'^* - \lambda^{*2} J^* J'\right)$$

let us write $|\Delta_1|^2$ using Eqn.(A-10e)

$$|\Delta_1|^2 = H'H'^* JJ^* + \frac{|q|^2|\lambda_0|^4}{|\gamma|^2 k^2 |\lambda|^4} HH^* J'J'^* +$$

$$+ \frac{iq^*\lambda_0^2}{\gamma^* k\lambda^{*2}} H^* J'^* H' J - \frac{iq\lambda_0^2}{\gamma k\lambda^2} HJ'H'^* J^*$$

and from Eqn.(A-37a) $|\phi_1|^2$ is given by

$$|\phi_1|^2 = H'H'^* JJ^* + \frac{|q|^2|\lambda_0|^4}{|\gamma|^2 k^2 |\lambda|^4} H^* HJ'J'^* +$$

$$+ \frac{iq^*\lambda_0^{*2}}{\gamma^* k\lambda^{*2}} HJ'^* H'^* J - \frac{iq\lambda_0^2}{\gamma k\lambda^2} H^* J' H' J'^*$$

Subtracting these two expressions and grouping the terms leads to

$$|\Delta_1|^2 - |\phi_1|^2 = \frac{4\lambda_0^2}{\pi a}\left[\frac{q^*}{\gamma^* k\lambda^{*2}} JJ'^* + \frac{q}{\gamma k\lambda^2} J^* J'\right] \qquad \text{(A-42b)}$$

where the relation $H'H^* - H'^* H = -\frac{4i}{\pi a}$ has been used since only the real values (propagating waves) of $(\lambda_0 r)$ are considered. Since

$$\gamma = i\sqrt{\frac{\varepsilon}{\mu}} \qquad \text{and} \qquad q = k\sqrt{\varepsilon\mu}$$

then

$$\frac{q}{\gamma k} = -i\mu \qquad \text{and} \qquad \frac{q^*}{\gamma^* k} = i\mu$$

then by substitution of these expressions into Eqn.(A-42b) leads to the wanted relation

$$|\Delta_1|^2 - |\phi_1|^2 = \frac{4i\lambda_0^2\mu}{\pi a|\lambda|^4}\left[\lambda^2 JJ'^* - \lambda^{*2}J^*J'\right] \qquad \text{(A-43)}$$

In a similar way it can be shown that

$$\phi_1\phi^* - \delta^*\Delta_1 = \frac{4ihn}{\pi k a^2}\left(1 - \frac{\lambda_0^2}{\lambda^2}\right)JJ^* \qquad \text{(A-44)}$$

and

$$\phi_1^*\phi - \delta\Delta_1^* = -\frac{4ihn}{\pi k a^2}\left(1 - \frac{\lambda_0^2}{\lambda^2}\right)JJ^* \qquad \text{(A-45)}$$

$A_{Pn}(h)$ becomes after substituting Eqns.(A-43), (A-44) and (A-45) into Eqn.(A-42)

$$A_{Pn}(h) = \frac{4i\lambda_0^2}{\pi a^2|\Delta|^2}\left\{\frac{a\mu}{|\lambda|^4}\left(|\Delta_2|^2 + |\delta|^2\right)\left(\lambda^2 JJ'^* - \lambda^{*2}J^*J'\right) + \right.$$

$$\left. + \frac{hn}{k}JJ^*\left[\left(\frac{1}{\lambda_0^2} - \frac{1}{\lambda^{*2}}\right)\left(\delta\Delta_1^* + \delta^*\Delta_2\right) - \left(\frac{1}{\lambda_0^2} - \frac{1}{\lambda^2}\right)\left(\delta^*\Delta_1 + \delta\Delta_2^*\right)\right]\right\} \qquad \text{(A-46)}$$

In a similar way the absorptivity of the cylinder to a $Q_0$ incident wave can be derived and the following expression is obtained

$$A_{Qn}(h) = \frac{4i\lambda_0^2}{\pi a^2|\Delta|^2}\left\{\frac{a\left(|\Delta_1|^2 + |\delta|^2\right)}{|\lambda|^4 k^2\mu}\left[h^2\left(\lambda^2 JJ'^* - \lambda^{*2}J^*J'\right) + \right.\right.$$

$$\left. + |\lambda|^4\left(JJ'^* - J^*J'\right)\right] + \frac{hn}{k}JJ^*\left[\left(\frac{1}{\lambda_0^2} - \frac{1}{\lambda^{*2}}\right)\left(\delta\Delta_2^* + \delta^*\Delta_1\right) - \right.$$

$$\left.\left. - \left(\frac{1}{\lambda_0^2} - \frac{1}{\lambda^2}\right)\left(\delta^*\Delta_2 + \delta\Delta_1^*\right)\right]\right\} \qquad \text{(A-47)}$$

Adding up Eqns.(A-46) and (A-47) and grouping some terms leads to

$$A_{Pn}(h) + A_{Qn}(h) = \frac{4i(\lambda^2 - \lambda^{*2})\lambda_a^2}{\pi k^2 a^2 |\Delta|^2 |\lambda|^4} \Bigg\{ \frac{2}{(\lambda^2 - \lambda^{*2})} \Big[ k^2\mu \left( |\Delta|^2 + |\delta|^2 \right) \times$$

$$\times \left(\lambda^2 J J'^{*} - \lambda^{*2} J^{*} J'\right) + \frac{1}{\mu} \left( |\Delta_1|^2 + |\delta|^2 \right) \Big( h^2 \left(\lambda^2 J J'^{*} - \lambda^{*2} J^{*} J'\right) +$$

$$+ |\lambda|^4 \left( J J'^{*} - J^{*} J' \right) \Big) \Big] - h n k J J^{*} \left[ \delta(\Delta_1^{*} + \Delta_2^{*}) + \delta^{*}(\Delta_1 + \Delta_2) \right] \Bigg\}, \qquad \text{(A-48)} \; \text{(R-IV.26)}$$

Comparing Eqn.(A-48) with the general solution (A-29) and noting that $\lambda^2 - \lambda^{*2} = q^2 - q^{*2} = k^2\mu(\varepsilon - \varepsilon^{*})$, it is then found that the power emitted by a cylinder can be expressed in terms of the absorptivity coefficients of the cylinder as follows:

$$P_{\omega} = \frac{2\pi^2 I_{\omega}}{k^2} \sum_{n=-\infty}^{+\infty} \int_{-k}^{+k} \left[ A_{Pn}(h) + A_{Qn}(h) \right] \frac{dh}{2\pi} \qquad \text{(A-49)} \; \text{(R-IV.27)}$$

## E - APPROXIMATE SOLUTIONS

Following Rytov's work we will now proceed to derive some approximate solutions for the equilibrium radiation emitted by a cylinder. These solutions are useful because they allow examining the influence of some parameters at limiting conditions and also, in some cases, allow testing of the theory by comparison with already established results.

Three limiting cases will be considered for which the radiating frequency $\omega$ is smaller than the plasma frequency $\omega_p$. This is the region of the frequency spectrum where the plasma essentially acts as a conductor. The three cases considered will correspond to a cylinder whose radius is smaller and larger than the skin depth while the skin depth is either large or small.

The Conducting Cylinder ($\omega_p^2/\omega^2 \gg 1$ and $\nu/\omega > 1$)

This case represents the low frequency approximation where the skin depth "d" is much smaller than the radius "a" of the cylinder. The skin depth is given by

$$d = \frac{C}{\sqrt{2\pi\mu\sigma\omega}} \qquad \text{(A-50)}$$

where $\sigma$ is the conductivity.

Since $\varepsilon = \varepsilon' - i\varepsilon''$ and in this case $\varepsilon' \ll \varepsilon''$ then the index of refraction is given by

$$n = \sqrt{\varepsilon\mu} = \sqrt{-i\varepsilon''\mu} = \sqrt{1 - \frac{i4\pi\sigma\mu}{\omega}} \qquad \text{(A-51)}$$

Substituting $\sigma$ from Eqn.(A-50) into (A-51) leads to

$$n = \sqrt{1 - \frac{2i}{k^2d^2}} \approx \sqrt{\frac{-2i}{k^2d^2}} = \frac{1}{kd}(1-i) \qquad \text{(A-52)}$$

Since $|\sqrt{\varepsilon\mu}| >> 1$ and $|h| << q$ then it can be assumed that

$$\lambda^2 = k^2\varepsilon\mu - h^2 \approx k^2\varepsilon\mu \qquad \text{(A-53)}$$

Thus $|\lambda|a$ is much greater than one and the following asymptotic expansions (McLachlan p. 83)[11] can be used.

$$J = J_n(\lambda a) = \sqrt{\frac{2}{\pi\lambda a}} \cos\left[\lambda a - \frac{(2n+1)\pi}{4}\right] \qquad \text{(A-54a)}$$

$$J' = \frac{dJn(\lambda a)}{da} = -\sqrt{\frac{2\lambda}{\pi a}} \sin\left[\lambda a - \frac{(2n+1)\pi}{4}\right] \qquad \text{(A-54b)}$$

Since from (A-52) and (A-53)

$$\lambda a = \frac{a}{d}(1-i)$$

it can then be shown that

$$\tan\left[\lambda a - \frac{(2n+1)\pi}{4}\right] \approx -i \qquad \text{(A-55a)}$$

and

$$\tan\left[\lambda^* a - \frac{(2n+1)\pi}{4}\right] \approx i \qquad \text{(A-55b)}$$

where the following relation is used

$$\tan(ia/d) = i \tanh(a/d) \approx i \quad \text{since} \quad a/d \gg 1$$

The following expressions can then be obtained using Eqns.(A-54a) and (A-54b)

$$\lambda^2 JJ'^* - \lambda^{*2} J^*J' = -\lambda^2 \frac{2}{\pi a}\sqrt{\frac{\lambda^*}{\lambda}} \sin\left[\lambda^* a - \frac{(2n+1)\pi}{4}\right]\cos\left[\lambda a - \frac{(2n+1)\pi}{4}\right] +$$

$$+ \lambda^{*2} \frac{2}{\pi a}\sqrt{\frac{\lambda}{\lambda^*}} \sin\left[\lambda a - \frac{(2n+1)\pi}{4}\right]\cos\left[\lambda^* a - \frac{(2n+1)\pi}{4}\right]$$

Multiplying and dividing through by

$$\cos\left[\lambda a - \frac{(2n+1)\pi}{4}\right]\cos\left[\lambda^* a - \frac{(2n+1)\pi}{4}\right]$$

and using Eqs.(A-55a) and (A-55b) the above expression becomes

$$\lambda^2 JJ'^* - \lambda^{*2} J^*J' = \frac{2}{i\pi a} |\lambda| (\lambda^* + \lambda) |\cos\alpha|^2 \qquad \text{(A-56a)}$$

where

$$\alpha = \lambda a - \frac{(2n+1)\pi}{4}$$

In a similar way it can be shown that

$$JJ'^* - J^*J' = \frac{2(\lambda + \lambda^*)}{i\pi a} |\cos\alpha|^2 \qquad \text{(A-56b)}$$

$$\Delta_1 = \sqrt{\frac{2}{\pi\lambda a}}\, H\rho \cos\alpha \qquad \text{(A-56c)}$$

$$\Delta_2 = \sqrt{\frac{2}{\pi\lambda a}}\, H\sigma \cos\alpha \qquad \text{(A-56d)}$$

$$\delta = \sqrt{\frac{2}{\pi\lambda a}}\, H\left(\frac{hn}{ka}\right) \cos\alpha \qquad \text{(A-56e)}$$

$$\Delta = \Delta_1\Delta_n - \delta^2 = \left[\rho\sigma - \left(\frac{hn}{ka}\right)^2\right]\left(\frac{2}{\pi\lambda a}\right) H^2\cos^2\alpha \qquad \text{(A-56f)}$$

$$|\Delta_2|^2 = \frac{2}{\pi a}\frac{|H|^2}{|\lambda|}|\sigma|^2\cos^2\alpha \qquad \text{(A-56g)}$$

$$|\delta|^2 = \frac{2}{\pi a}\frac{|H|^2}{|\lambda|}\left(\frac{hn}{ka}\right)^2\cos^2\alpha \qquad \text{(A-56h)}$$

$$|\Delta| = \left|\rho\sigma - \left(\frac{hn}{ka}\right)^2\right|^2 \frac{4}{\pi^2 a^2}\frac{|H|^4}{|\lambda|^2}\cos^4\alpha \qquad \text{(A-56i)}$$

where

$$\rho = \frac{H'}{H} + \frac{\mu\lambda_a^2}{i\lambda} \quad ; \quad \sigma = \frac{H'}{H} + \frac{\varepsilon\lambda_a^2}{i\lambda} \qquad \text{(A-57)}$$

The equations (A-56) are then introduced into Eqn.(A-48) which becomes, after using the relations $\lambda^2 - \lambda^{*2} = k^2\mu(\varepsilon - \varepsilon^*)$; $\lambda + \lambda^* = k\sqrt{\mu}\,(\sqrt{\varepsilon} + \sqrt{\varepsilon^*})$

$$A_{\cdot}(h) + A_{Qn}(h) = \frac{4i\lambda_a^2(\varepsilon - \varepsilon^*)}{\pi k^2 a|\varepsilon|\;|H|^2}\frac{1}{|\rho\sigma - (\frac{hn}{ka})^2|^2}\times$$

$$\times\left\{\frac{k\sqrt{\mu}\,(\sqrt{\varepsilon} + \sqrt{\varepsilon^*})}{i(\varepsilon - \varepsilon^*)}\left[|\sigma|^2 + \left(\frac{hn}{ka}\right)^2 + \frac{|\varepsilon|}{\mu}\left(|\rho|^2 + \left(\frac{hn}{ka}\right)^2\right)\right]\right.$$

$$\left. - \left(\frac{hn}{ka}\right)^2\frac{(\rho + \rho^* + \sigma + \sigma^*)}{|\varepsilon|\mu}\right\} \qquad \text{(A-58)}$$

1. Thin, Wall Conducting Cylinder $(ka \ll 1)\ (|\lambda| a \gg 1)$

In this case, since $\lambda_o a \ll 1$ (McLachlan, p.26)[11]

$$H = H_n(\lambda_o a) \approx i N_n(\lambda_o a) \approx \frac{i(n-1)!}{\pi}\left(\frac{2}{\lambda_o a}\right)^n$$

and

$$H' = H_n'(\lambda_o a) = -\frac{i n!}{\pi a}\left(\frac{2}{\lambda_o a}\right)^n$$

and

$$H_o(\lambda_o a) = -\frac{2i}{\pi}\ln\left(\frac{\lambda_o a}{2}\right) \qquad \text{(A-59)}$$

$$H_o'(\lambda_o a) = -\frac{2i}{\pi a}$$

Let us examine the variation of $A_{Pn}(h) + A_{Qn}(h)$ (Eqn.A-58) in terms of n. Since from (A-59) $H'/H$ is proportional to n, then the product $\rho\sigma \sim n^2$ and $\left|\frac{1}{\rho\sigma - (hn/ka)^2}\right|^2 \sim n^{-4}$.

On the other hand the terms inside the curly bracket (A-58) can be shown to be proportional to $n^4$ and thus cancel the n variation of the factor $|\rho\sigma - (hn/ka)^2|^2$. The expression (A-58) also involves the factor $1/|H|^2$ which is dependent on n. From (A-59), one finds that $1/|H|^2 \approx (\lambda_o a)^{2n}$. Therefore $A_{Pn}(h) +$ $+ A_{Qn}(h) \sim (\lambda_o a)^{2n}$.

Since $(\lambda_o a) \ll 1$, one can retain only the first term in the series expansion Eqn.(A-29) for the emitted power $P_\omega$ and Eqn.(A-58) becomes with $n = 0$

$$A_{P_o}(h) + A_{Q_o}(h) = \frac{4\lambda_o^2 \sqrt{\mu}\,(\sqrt{\varepsilon} + \sqrt{\varepsilon^*})}{\pi k a |\varepsilon|\, |N_o|^2} \left[ \frac{1}{|\rho|^2} + \frac{|\varepsilon|}{\mu |\sigma|^2} \right] \qquad \text{(A-60)}$$

Introducing Eqn.(A-59) into Eqn.(A-57) and using the relations (A-52) and (A-53) one then obtains for $\rho$ and $\sigma$ the following expressions:

$$\rho = \frac{1}{a \ln\left(\frac{\lambda_o a}{2}\right)} \left[ 1 - \frac{i\lambda_o^2 a \mu d \ln\left(\frac{\lambda_o a}{2}\right)}{(1-i)} \right] \qquad \text{(A-61a)}$$

$$\sigma = \frac{1}{a \ln\left(\frac{\lambda_o a}{2}\right)} \frac{(1-i)}{k\mu d} \left[ \frac{k\mu d}{(1-i)} - \frac{i\lambda_o^2 a \ln\left(\frac{\lambda_o a}{2}\right)}{k} \right] \qquad \text{(A-61b)}$$

With the help of (A-59) and (A-52) the coefficient in front of the square bracket in (A-60) becomes

$$\frac{4\lambda_o^2 \sqrt{\mu}\,(\sqrt{\varepsilon} + \sqrt{\varepsilon^*})}{\pi k a |\varepsilon|\, |N_o|^2} = \frac{\pi\lambda_o^2 \mu a d}{a^2 \left[\ln\left(\frac{\lambda_o a}{2}\right)\right]^2} \qquad \text{(A-62)}$$

Eqns.(A-61) and (A-62) are then substituted into Eqn.(A-60) and one finds

$$A_{P_o}(h) + A_{Q_o}(h) = \pi\lambda_o^2 a \mu d \left\{ \left| 1 - \frac{i\lambda_o^2 a \mu d}{(1-i)} \ln\left(\frac{\lambda_o a}{2}\right) \right|^{-2} + \right.$$

$$\left. + \left| \frac{k\mu d}{(1-i)} - \frac{i\lambda_o^2 a}{k} \ln\left(\frac{\lambda_o a}{a}\right) \right|^{-2} \right\} \qquad \text{(A-63)}$$

Substituting (A-63) into Eqn.(A-29) and changing variables as follows: $h = k \sin\theta$ , $\lambda_o = \sqrt{k^2 - h^2} = k\cos\theta$ , one arrives at

$$P_\omega = 2\pi^2 k\, a\, \mu\, d\, I_{o\omega} \int_0^{\pi/2} \left\{ \left| 1 - \frac{i k^2 a \mu d}{(1-i)} \cos^2\theta \ln\left(\frac{k a \cos\theta}{2}\right) \right|^{-2} + \right.$$

$$\left. + \left| \frac{k \mu d}{(1-i)} - i k a \cos^2\theta \ln\left(\frac{k a \cos\theta}{2}\right) \right|^{-2} \right\} \cos^3\theta \, d\theta \qquad \text{(A-64)}$$

It can be shown, using l'Hopital's rule, that the product $\cos^2\theta \ln(\frac{k a \cos\theta}{2})$ in Eqn.(A-64) tends to zero at the limit $\theta = \pi/2$. From the conditions $ka \ll 1$ and $d/a \ll 1$, it is thus evident that

$$\left| \frac{k^2 a \mu d}{1-i} \cos^2\theta \ln\left(\frac{k a \cos\theta}{2}\right) \right| \ll 1$$

Therefore this term is neglected in front of one in Eqn.(A-64). Since $\ln(\frac{k a \cos\theta}{2})$ is a slowly varying function of $\theta$, the $\theta$ variation is neglected inside the logarithm. Using the notation $\alpha = k a \ln(\frac{ka}{2})$, it follows that Eqn.(A-64) reduces to

$$P_\omega = 2\pi^2 k\, a\, \mu\, d\, I_{o\omega} \int_0^{\pi/2} \left\{ 1 + \frac{1}{\left[ \frac{k^2 \mu^2 d^2}{2} - k \mu d \alpha \cos^2\theta + \alpha^2 \cos^4\theta \right]} \right\} \cos^3\theta \, d\theta$$

(A-65)

The first term of the integral gives 2/3, and the integration of the second term leads to a result different from that given by Rytov. We will then proceed in the following to show the procedure taken in evaluating this integral.

Consider the evaluation of the following integral:

$$I = \int_0^{\pi/2} \frac{\cos^3\theta \, d\theta}{\alpha^2\cos^4\theta - \alpha(k\mu d)\cos^2\theta + \frac{(k\mu d)^2}{2}}$$

(Page 234 of Rytov's monograph[1] .)

We let $\beta = \frac{k\mu d}{-\alpha}$ since $\alpha = ka \ln \frac{ka}{2} < 0$. The constant $\beta \lll 1$.

The integral becomes:

$$I = \frac{1}{\alpha^2} \int_0^{\pi/2} \frac{\cos^3\theta \, d\theta}{\cos^4\theta + \beta\cos^2\theta + \frac{\beta^2}{2}}$$

On finding the roots of the denominator, which we denote by a, we can write I in the following form:

$$I = \frac{1}{\alpha^2 A}\left[\int_0^{\pi/2} \frac{\cos^2\theta \, d\theta}{\cos^2\theta - A\cos\theta + B} - \int_0^{\pi/2} \frac{\cos^2\theta \, d\theta}{\cos^2\theta + A\cos\theta + B}\right]$$

where:

$$A = a + a^*$$

$$B = a a^*$$

$$a = \sqrt{\frac{|\beta|}{\sqrt{2}}}\; e^{i\frac{3\pi}{8}}$$

If we now let $y = \cos\theta$ we have

$$I = \frac{1}{\alpha^2}\left[\int_0^1 \frac{y\,dy}{\sqrt{1-y^2}\,[y^2 - Ay + B]} + \int_0^1 \frac{y\,dy}{\sqrt{1-y^2}\,[y^2 + Ay + B]} - \right.$$

$$\left. - \frac{B}{A}\left(\int_0^1 \frac{dy}{\sqrt{1-y^2}\,[y^2 - Ay + B]} - \int_0^1 \frac{dy}{\sqrt{1-y^2}\,[y^2 + Ay + B]}\right)\right]$$

To perform these integrals we make use of the following substitution:

$$y = \frac{k_1 t + k_2}{A(t+1)}$$

where:

$$k_1 = (1+B) + \sqrt{(1+B)^2 - A^2}$$

$$k_2 = (1+B) - \sqrt{(1+B)^2 - A^2}$$

In this way we find:

$$I = \frac{A(k_1 - k_2)}{a^2 k_3 \cdot \sqrt{-k_5}} \left[ (k_1 - B) \int_{-\frac{k_2}{k_1}}^{\frac{-A+k_2}{+A-k_1}} \frac{t\,dt}{D(t)} + (k_2 - B) \int_{-\frac{k_2}{k_1}}^{\frac{A+k_2}{-A-k_1}} \frac{dt}{D(t)} + \right.$$

$$\left. + (k_1 + B) \int_{-\frac{k_2}{k_1}}^{\frac{-A+k_2}{+A-k_1}} \frac{t\,dt}{D(t)} + (k_2 + B) \int_{-\frac{k_2}{k_1}}^{\frac{+A+k_2}{-A-k_1}} \frac{t\,dt}{D(t)} \right].$$

where

$$D = \left[ \frac{k_4}{k_3} + t^2 \right] \sqrt{\frac{k_4}{-k_5} - t^2} = (c^2 + t^2)\sqrt{a^2 - t^2}$$

$$k_3 = k_1^2 - A^2 k_1 + A^2 B \qquad c^2 = \frac{k_4}{k_3}$$

$$k_4 = k_2^2 - A^2 k_2 + A^2 B$$

$$k_5 = A^2 - k_1^2 \qquad a^2 = \frac{k_4}{-k_5}$$

$$k_6 = A^2 - k_2^2$$

By means of the substitution:

$$u = \frac{1}{\sqrt{a^2 - t^2}}$$

we can write

$$\int \frac{t\,dt}{(c^2 + t^2)\sqrt{a^2 - t^2}}$$

in the form

$$\int \frac{du}{[(a^2 + c^2)u^2 - 1]} = \frac{1}{2\sqrt{a^2 + c^2}} \log \frac{\sqrt{a^2 + c^2}\,u - 1}{\sqrt{a^2 + c^2}\,u + 1}$$

The integral

$$\int \frac{dt}{(c^2 + t^2)\sqrt{a^2 - t^2}}$$

can be written as (page 55, no. 20 of integral tables[13]):

$$\frac{1}{c} \frac{1}{\sqrt{c^2 + a^2}} \tan^{-1} \frac{\sqrt{a^2 + c^2}\,t}{c\sqrt{a^2 - t^2}} + \text{constant}$$

By using the following identities

$$A^2 = k_1 k_2$$

$$k_6 = A^2 - k_2^2 = k_1 k_2 - k_2^2 = k_2(k_1 - k_2) > 0$$

$$k_5 = A^2 - k_1^2 = k_1 k_2 - k_2^2 = -k_1(k_1 - k_2) < 0$$

$$\frac{k_6}{-k_5} = \frac{k_2}{k_1}$$

$$B = \frac{(k_1 + k_2)}{2} - 1$$

$$k_4 = k_2^2 - A^2 k_2 + A^2 B = \frac{k_2(k_1 - k_2)(k_1 - 2)}{2}$$

$$k_3 = k_1^2 - A^2 k_1 + A^2 B = - \frac{k_1(k_1 - k_2)(k_2 - 2)}{2}$$

Expressing these quantities in terms of $\beta$ (using the fact $\beta \ll 1$) and substituting in the integrals we find: (keeping only the largest term):

$$I \approx - \frac{1}{2\alpha\pi}\left(1 + \frac{\beta}{2\sqrt{2}}\right) \log \frac{\beta \cos^2 \pi/8}{\sqrt{2} \cdot 4} \qquad \text{where } (\beta \ll 1)$$

$$\approx - \frac{1}{2\alpha\pi} \log \beta = - \frac{1}{2\alpha\pi} \log \frac{k\pi d}{|\alpha|}$$

2. Well Conducting Cylinder With Large Radius $(ka \gg 1)$

The basic conditions are then:

$$ka \gg 1$$

$$|\lambda| a = \sqrt{2} a/d \gg 1 \qquad (a \gg d) .$$

When $|n| \ll ka$, we can represent the Hankel functions by the following approximate expansions (McLachlan, p. 198)[11]

$$H_n^2(\lambda_0 a) = \sqrt{\frac{2}{\pi \lambda_0 a}}\, e^{-i[\lambda_0 a - \frac{(2n+1)\pi}{4}]}$$

$$H_n^{2\prime}(\lambda_0 a) = - i \sqrt{\frac{2\lambda_0}{\pi a}}\, e^{-i[\lambda_0 a - \frac{(2n+1)\pi}{4}]} \qquad \text{(A-67)}$$

When $|n| \gg ka$, the Hankel function is given by

$$H_n^2(\lambda_e a) \sim i\left(\frac{2}{\lambda_e a}\right)^n \frac{(n-1)!}{\pi} \; .$$

Thus, because of the coefficient $\frac{1}{|H|^2}$ in Eqn.(A-58)

$$A_{Pn}(h) + A_{Qn}(h) \sim \frac{1}{\left(\frac{2}{\lambda_e a}\right)^{2n}\left[(n-1)!\right]^2} \to 0 \quad \text{for large } |n| .$$

It is then assumed that the values of n extend only from $-\chi k a$ to $+\chi k a$, where $\chi$ is a coefficient of order 1.

Combining Eqn.(A-67) with Eqn.(A-57) and using Eqns.(A-52) and (A-53), one obtains

$$\begin{aligned} \rho &= -i\lambda_e + \frac{\mu\lambda_e^2}{i\lambda} \approx -i\lambda_e \\ \sigma &= -i\lambda_e + \frac{\varepsilon\lambda_e^2}{i\lambda} \approx \frac{\varepsilon\lambda_e^2}{\lambda} \end{aligned} \qquad \text{(A-68)}$$

Substituting (A-67) and (A-68) into (A-58) gives

$$A_{Pn}(h) + A_{Qn}(h) = \frac{2i\lambda_e^3(\varepsilon-\varepsilon^*)}{k^2\,|\varepsilon|}\;\frac{1}{\left|\frac{\varepsilon\lambda_e^2}{\lambda} + \left(\frac{hn}{ka}\right)^2\right|^2} \times$$

$$\times\left\{\frac{k\sqrt{\mu}\,(\sqrt{\varepsilon}+\sqrt{\varepsilon^*})}{i(\varepsilon-\varepsilon^*)}\left[\frac{|\varepsilon|^2\lambda_e^4}{|\lambda|^2} + \left(\frac{hn}{ka}\right)^2 + \frac{|\varepsilon|}{\mu}\left(\lambda_e^2 + \left(\frac{hn}{ka}\right)^2\right)\right] - \right.$$

$$\left. -\left(\frac{hn}{ka}\right)^2 \frac{i\lambda_e^2}{|\varepsilon|\mu}\left(\frac{\varepsilon^*}{\lambda^*} - \frac{\varepsilon}{\lambda}\right)\right\}$$

Retaining only the terms containing higher power of $\sqrt{\varepsilon}$, this equation becomes, with the help of $|\lambda|^2 = k^2\mu|\varepsilon|$

$$A_{Pn}(h) + A_{Qn}(h) = \frac{2}{k}\left(\sqrt{\frac{\varepsilon}{\mu}} + \sqrt{\frac{\varepsilon^*}{\mu}}\right)\lambda_0^3 \frac{\left[\frac{\lambda_0^4}{k^2} + \lambda_0^2 + \left(\frac{hn}{ka}\right)^2\right]}{\left|\frac{\varepsilon\lambda_0^2}{\lambda} + \left(\frac{hn}{ka}\right)^2\right|^2} \qquad \text{(A-69)}$$

The following definitions are then introduced

$$h = k\sin\theta \quad ; \quad \lambda_0 = k\cos\theta \quad ; \quad y = n/ka$$

and the summation in Eqn.(A-29) is changed to an integration as follows

$$\sum_{n=-\chi ka}^{+\chi ka} \rightarrow ka\int_{-\chi}^{+\chi} dy$$

Applying these definitions to Eqn.(A-29), one gets

$$P = 4\pi a I_{o\omega}\int_0^{\chi}\int_0^{\pi/2} [\, A_P(y,\theta) + A_Q(y,\theta)\,]\, dy \cos\theta \, d\theta \qquad \text{(A-70)}$$

and Eqn.(A-69) becomes

$$A_P(y,\theta) + A_Q(y,\theta) = 2\left(\sqrt{\frac{\varepsilon}{\mu}} + \sqrt{\frac{\varepsilon^*}{\mu}}\right) \frac{[\cos^4\theta + \cos^2\theta + y^2\sin^2\theta]\cos^3\theta}{\left|\sqrt{\frac{\varepsilon}{\mu}}\cos^3\theta + y^2\sin^2\theta\right|^2} \qquad \text{(A-71)}$$

Substituting Eqn.(A-71) into Eqn.(A-70) leads to

$$P_\omega = 8\pi k^2 a I_{o\omega}\left(\sqrt{\frac{\varepsilon}{\mu}} + \sqrt{\frac{\varepsilon^*}{\mu}}\right)\int_0^{\chi} dy \int_0^{\pi/2} \frac{[\cos^4\theta + \cos^2\theta + y^2\sin^2\theta]\cos^4\theta \, d\theta}{\left|\sqrt{\frac{\varepsilon}{\mu}}\cos^3\theta + y^2\sin^2\theta\right|^2} \qquad \text{(A-72)}$$

Since y varies from 0 to $\chi \sim 1$, then with the same precision as Eqn.(A-72) was obtained, one can neglect terms in $y^2$. Eqn.(A-72) can thus be integrated and the following expression is obtained

$$P_\omega = 6\pi^2 \chi a \frac{\sqrt{\mu}(\sqrt{\varepsilon} + \sqrt{\varepsilon^*})}{|\varepsilon|} I_{e\omega}$$

Using Eqn.(A-52), one obtains

$$P_\omega = 6\pi^2 \chi a I_{e\omega} k \mu d$$

and using the Rayleigh-Jeans approximation for $I_{e\omega}$, one finally gets

$$P_{s\omega} = \frac{P_\omega}{2\pi a} = \frac{3\chi K T}{4\pi^2} k^3 \mu d$$

In another problem (a slab in a waveguide) Rytov has shown that $\chi = 8/9$ and therefore

$$P_{s\omega} = \frac{2K T}{3\pi^2} k^3 \mu d \qquad \text{(A-73)}$$

3. Thin, Poorly Conducting Cylinder

In this case, the following conditions apply

$ka << 1$ and $a << d$

or

$\lambda_e a << 1$ and $|\lambda a| << 1$

It is thus sufficient to keep only the first term in the series

expansion of the Hankel and the Bessel functions, i.e. the $H_n^{(2)}$ is given by Eqn.(A-59) and $J_n$ is given by

$$J_0(\lambda a) = 1 \quad ; \quad J_n(\lambda a) = \frac{1}{n!}\left(\frac{\lambda a}{2}\right)^n$$

$$J_0'(\lambda a) = -\frac{\lambda^2 a}{2} \quad ; \quad J_n'(\ a) = \frac{1}{(n-1)!}\left(\frac{\lambda a}{2}\right)^n \tag{A-74}$$

Since Eqn.(A-67) is again proportional to $(\lambda_0 a)^{2n}$, (same argument as for the thin well conducting cylinder) one can keep only the first term (Eqn.A-29) in the summation over n, i.e. n = 0. The following relations are then obtained:

$$\lambda^2 JJ'^* - \lambda^{*2} J^* J' = 0$$

$$JJ'^* - J^* J' = \frac{(\lambda^2 - \lambda^{*2})a}{2} \tag{A-75}$$

$$\Delta_1 = \Delta_2 = -\frac{2i}{\pi a} \quad ; \quad \delta = 0 \quad ; \quad \Delta = \frac{4}{\pi^2 a^2}$$

Substituting Eqns.(A-75) into Eqn.(A-48) gives

$$A_{P_0}(h) + A_{Q_0}(h) = \frac{i\pi a^2 \lambda_0^2 (\lambda^2 - \lambda^{*2})}{2k^2 \mu} \tag{A-76}$$

Using the definitions for $\lambda^2$ and $\lambda_0^2$ one gets

$$A_{P_0}(h) + A_{Q_0}(h) = \pi a^2 \varepsilon''(k^2 - h^2) \tag{A-77}$$

Eqn.(A-77) is then introduced into Eqn.(A-29) and the following expression is obtained:

$$P_\omega = \frac{2\pi^2 a^2 \epsilon''}{k^2} I_{e\omega} \int_0^k (k^2 - h^2)\,dh = \frac{4\pi^2 k a^2 \epsilon''}{3} I_{e\omega} \,.$$

The expression for the power emitted per unit area of the cylinder, using the Rayleigh-Jeans law, thus becomes

$$P_{a\omega} = \frac{P_\omega}{2\pi a} = \frac{(KT) k^3 a \epsilon''}{6\pi^2} \qquad \text{(A-78)}$$

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